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Doctoral Thesis

Sangaku: A Mathematical, Artistic, Religious, and Diagrammatic Examination

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Declaration of Authorship

I, Rosalie Hosking, declare that this thesis titled, 'Sangaku: A Mathematical, Artistic, Religious, and Diagrammatic Examination' and the work presented in it are my own. I confirm that:

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- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
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Abstract

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by Rosalie HOSKING

Between the 17th and 19th centuries, mathematically orientated votive tablets appeared in Shinto shrines and Buddhist temples all over Japan. Known as *sangaku*, they contained problems of a largely geometrical nature.

The 18th century also saw the Japanese mathematician Seki Takakazu develop a form of algebra known as *tenzan jutsu*. To date, much of the literature regarding *sangaku* employs modern mathematical notation and techniques to solve their problems. In this research I solve problems from nine *sangaku* using modern techniques. As well as this, I also illustrate how *sangaku* can be solved using the traditional Japanese *tenzan jutsu* of Seki. I compare seven problems taken from *sangaku* with similar problems solved using *tenzan jutsu* from the 1810 Japanese text the *Sanpo Tenzan Shinan*. I show the *tenzan jutsu* technique can be applied to solve the *sangaku* problems.

The question of whether these tablets had purely communicative and mathematical functions is also still debated by researchers. In this thesis I argue that an examination of the creation, location, and writings on these tablets by their authors shows *sangaku* had multiple functions and should be considered artistic, religious, as well as mathematical artefacts. I also examine the role of diagrams on *sangaku*, and argue that they are a vital element of determination for the problems on these tablets.

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Lastly I would like to mention my friends and family who stood by me during this research and gave me their unfailing support. I wish to recognise my most enthusiastic supporters, my Mother and late Father, who provided endless encouragement.

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Definitions

Edo Period	Japanese era from 1603 to 1868 CE
Tokuagawa Period	Alternative name for Edo period
Meiji Period	Japanese era from 1868 to 1912 CE
Wasan	Traditional Edo period Japanese Mathematics
Sangaku	Edo period Votive Mathematical Tablets
Ema	Small votive tablet found in Japanese shrines and temples
Ōema	Large votive tabet found in Japanese shrines and temples
Hōnō	Offering of worship to relgious location
Emadokoro	Hall cotaining ema and *ōema*
Shinto	Native Japanese religion
Kanbun	Japanese academic language similar to Chinese
Bōsho Hō	Traditional Japanese side writing technique
Tenzan Jutsu	Traditional Japanese symbolic manipulation technique
Hachiman	Japanese deity of protection and war
Tenjin	Japanese deity of academia
Inari	Japanese fox deity
Soroban	Japanese bead abacus
Sangi	Japanese rod abacus
Iroha	Japanese pangram
Hinoki	Popular Japanese cedar or cypress
Sumi	Japanese paint made from soot
Enogu	Japanese traditional paints, usually made from pigments
Ameterasu	Female Japanese deity of the sun

Dedicated to my father Lindsay Bernard Hosking (1954-2014), who supported me and so dearly wanted to see the fruits of this research. May you rest in peace.

Chapter 1

Introduction

Amongst the dusty, faded plaques of an ancient Kyoto shrine lies a forgotten treasure. Though the years have not been kind, the brilliant shapes and colours so masterfully laid on its dry wooden surface still maintain enough of their original glory to catch the eye of the perceptive observer. This wooden relic – seeming to blend religion, art, and mathematics – is known as a *sangaku* 算額.

Sangaku, where *san* 算 means calculation and *gaku* 額 plaque, are wooden tablets containing mathematics. Created during the Edo (1603-1868 CE) and Meiji (1868 - 1912) periods, they are found in Shinto shrines and Buddhist temples throughout Japan. They presented mathematical problems looking at the geometry of certain figures and the relationships between them. They ranged in degree of difficulty, and their subject matter included results similar to the Malfatti theorem, Casey theorem, and Soddy hexlet theorem, which appeared on *sangaku* prior to being known in Europe [71, p. 85]. A typical *sangaku* however deals with finding the diameter of circles, the side lengths of triangles, squares, and hexagons, and the relationships between different figures. These tablets could contain anywhere from one to twenty problems, and could be over a metre in height and width.

1.1 Literature Review

Sangaku were first introduced to English language readers in David Eugene Smith and Yoshio Mikami's *A History of Japanese Mathematics* of 1914. This was the first detailed English text on the history of mathematics in Japan, covering the development of mathematics from the year 500 CE through to the introduction of Western mathematics in the late 1800s. Here while discussing the mathematician Fujita Sadasuke, they mention a text containing problems from various *sangaku*. The tablets, not yet referred to as *sangaku*, are introduced as "problems that had been hung before various temples by certain mathematical devotees" [87, p. 184]. Smith and Mikami theorised that they may have been created due to "a desire for the praise or approval of the gods", or as a way for mathematicians to post challenges to one another similar to how "European students in the Middle Ages would post a thesis on the door of a church" [87, 184]. Though Smith and Mikami did not dwell further on the tablets, their work brought this interesting practice of Japanese mathematics to light, leaving the door open for further research into why these tablets were created, who created them, and why they were located in Shinto shrines and Buddhist temples.

FIGURE 1.1: A *sangaku* from Shoganji temple, Nagano prefecture. (Image by author).

In the 1970's, an increasing number of mathematicians in Japan became interested in documenting the history of mathematics of their country. They formed societies of native Japanese mathematics and began to publish books on the subject. However *sangaku* were not the focus. For example 'The Mathematics of the Japanese' *Nihonjin no sūgaku* 日本人の数学 by Shimodaira Kazuo, published in 1972, only discusses them in relation to mathematicians, with no mention of the history and nature of the tablets. This changed in 1976 with the appearance of the 'Collection of Aichi Prefecture Sangaku' *Aichiken sangaku shū* 愛知県算額集 by Hidetoshi Fukagawa, the first book solely dedicated to examining *sangaku* and their problems. Fukagawa, a school teacher with a PhD in mathematics, came across mention of *sangaku* in a history book [71, p. 85]. He sought out these tablets, and began to study them and their problems. Fukugawa followed this work with 'Collection of Aichi Prefecture Sangaku - Continued' *Aichi ken sangaku shū zoku* 愛知県算額集続 in 1980. While initially focusing on tablets of his local Aichi prefecture, Fukagawa later collaborated with Dan Pedoe to produce the English language *Japanese Temple Geometry Problems: San Gaku*, which presented *sangaku* problems from all over Japan. This work was the first English book focused on *sangaku*. It presented a large number of problems from the tablets, along with solutions in modern mathematical notation. Fukagawa and Pedoe also reflected on the history and function of the tablets, concluding they were created by "many skilled geometers who wished to thank the god or gods for the discovery of a particularly lovely theorem, and...who were not averse to displaying their discoveries for other geometers" [24, p. i].

From the early 1900s to the 1990s, it was this view of *sangaku* - considering them as proclamations of thanks to the gods and a way to publish mathematics - that became mainstream. However, this conception started to change in the 1990s and 2000s when their format as religious offerings was contended to be only a front, with the real purpose of the tablets being the self promotion of the author and mathematical communication with other practitioners.

The first literature to dismiss the idea that *sangaku* functioned as religious offerings was the 1998 French language article *Les mathématiques peuvent-elles n'être que pour divertissement? Une analyse des tablettes votives de mathématiques à l'époque d'Edo* by Annick Horiuchi. Horiuchi, a specialist in the area of Japanese mathematics, published the standard text for Edo mathematics *Japanese Mathematics in the Edo Period (1600-1868): A study of the works of Seki Takakazu (?-1708) and Takebe Katahiro (1664-1739)* first in French in 1994, and then into English in 2001. In her article on the tablets, Horiuchi contends that *sangaku* were in no way offerings or dedications to the gods, being instead "part of a specific historical context marked by increased diffusion of the discipline in the country, the professionalization of the masters...and then by the strong competition between schools. The tablets stand out as instruments of communication" [31, p. 145]. Her argument - based off an investigation of primary sources, the wider mathematical history, and its context - showed top Edo mathematicians were often producing, reading, considering, or solving *sangaku*. She promoted the idea of seeing the tablets as a way these competitive mathematicians and schools in the Edo period could communicate given the absence of journals and universities. The tablets were a means for authors to publish and spread their mathematical discoveries and results, and gain fame and recognition.

Horiuchi's view of *sangaku* began to be adopted by other historians from the mid 1990s onwards. In Dennis Normile's article *"Amateur" Proofs Blend Religion and Scholarship in Ancient Japan* from 2005, although the title points to a connection between religion and mathematics in Japan, Fukagawa is quoted as stating "Ostensibly, the tablets were left as gifts to the gods....In reality people were showing off and challenging others to work out the proof" [62, p. 1716]. Here Fukagawa's conception of *sangaku* aligned more with Horiuchi, seeing *sangaku* as on the surface having the physical appearance of religious offerings, but having challenges and self promotion as the real purpose of the mathematician. Others began to agree with this view, such as Fukuzo Suzuki who writes in 2001 in *An Equilateral Triangle with Sides through Vertices of an Isosceles Triangle* that "Results and theorems of traditional Japanese mathematics, known as

Wasan, were usually stated in the form of problems; these were originally displayed on wooden tablets (Sangaku) hung in shrines and temple" [92, p 304]. Suzuki's description of *sangaku* makes no mention of any religious element or function as offerings to the gods. Instead, his statement suggests that they functioned as a publishing device and means to publicly display results.

It was during this time that *sangaku* became popularised in Tony Rothman's article *Japanese Temple Geometry*, published in the May 1998 edition of *Scientific American*. Rothman introduced *sangaku* once again to a Western audience, and the large reader base of his medium saw the tradition reach a number of historians outside Japan. Rothman produced this article with the cooperation of Fukagawa, and in it describes the known history of the tablets and the type of mathematics they contain. He states that "It is natural to wonder who created the *sangaku* and when, but it is easier to ask such questions than to answer them" [71, p. 85]. In the joint 2008 work *Sacred Geometry: Japanese Temple Mathematics* by Fukagawa and Rothman, some of these questions began to be answered in more detail. Similar to Fukagawa's work with Pedoe, the text provided many *sangaku* problems and their solutions. They also gave a detailed history of the tradition and Japanese mathematics in general, dwelling more on the question of who made the tablets and when. They included the travel diary of the mathematician Yamaguchi Kanzan, who roamed Japan studying and creating *sangaku*, and translated some prefaces to *sangaku* which provided readers with details from the authors themselves about their tablets.

In his work with Fukagawa, Tony Rothman argues against the idea that *sangaku* were merely forms of mathematical communication, writing "Japanese mathematicians often hung the tablets as acts of worship, thanks to the gods for being able to solve a difficult problem" [29, p. xvi-xvii]. They were, according to Rothman, a blend of worship and mathematics that can be described as "sacred mathematics" [29, p. xvii]. It was around this time in the 2000s that the idea of *sangaku* as mathematical offerings began to re-emerge. For example, Freeman Dyson called these tablets "a work of art as well as a mathematical statement" [29, p. x] in 2008. Also, in their 2005 work *Circle Packings and the Sacred Lotus* Tarnai and Miyazaki write "A mathematician having made an interesting discovery would write it down on a wooden board called a *sangaku*, dedicated it to the gods or Buddha and hang it up under the roof of a shrine or temple" [94, p. 148], presenting a position that these tablets were dedicated to Japanese deities. As well as this, in his 2012 article *Historical Notes: Sangaku - The Mathematics of Traditional Japanese Votive Tablets*, when discussing the purpose of *sangaku*, Albrecht Heeffer states the "Religious and spiritual purpose is certainly one of these" [28, p. 277]. He remarks on the journey of the mentioned mathematician Yamaguchi Kanzan as being as much a spiritual pilgrimage as it was mathematical research, and notes how these tablets fit with the aesthetic principles of shrines [28, p. 277].

The conception of these tablets as forms of art also began to achieve wider acceptance. For example, in a recent 2014 article in collaboration with Kazunori Horibe, *Sangaku - Japanese Mathematics and Art in the 18th, 19th, and 20th Centuries*, Fukagawa - while not stating the tablets are religious offerings - describes them as being "presented as works of art" [23, p. 111]. Included in this article is a section on *sangaku* as art, where they show "some beautiful sangaku which are mainly artistic works rather than mathematics" [23, p. 115]. This indicates that in recent years a shift has occurred towards the conception that the tablets functioned as objects of religious worship and art. However, not all authors agree. For instance, Miroslaw Majewski, Jen-Chung Chaun, and Nishizawa Hitoshi in their 2010 article *The New Temple Geometry Problems in Hirotaka's Ebisui Files* maintain the position that these tablets functioned as "a kind of challenge for other people who attended the shrine – 'look I proved this, I am a clever person, can you also prove it?"' [48, p. 1].

It can be seen that in the literature the *sangaku* tradition has been much explored in the last century. Historians have worked on understanding and solving their mathematical problems, and discussed - though not always in agreement - their purpose. However, there are still some areas yet to be explored in the literature. One such area is mentioned by Peter J Lu in his 2008 review of Fukagawa and Rothman's *Sacred Geometry: Japanese Temple Mathematics* titled *The Blossoming of Japanese Mathematics*. Here Lu expresses dissatisfaction with the presentation of Edo period mathematics only in terms of modern mathematics, feeling that "Illustrating traditional Japanese and modern Western methods side by side would have been instructive" [47, p. 1050]. Lu believes expressing the work with modern methods "achieves only limited success in showcasing *sangaku* as exemplars of a uniquely Japanese style of mathematics, because that style is never elucidated" [47, p. 1050]. Rothman and Fukagawa's book presents *sangaku* on a level accessible to the non-mathematician interested in history, and due to this keeps the language and content within reach of a broader audience. Other works which have followed also present the problems using modern mathematics. Two examples are the 2009 Portuguese text *Sangaku* by Antonieta Constantino, which provides solutions to *sangaku* problems using modern mathematical notation, and the 2010 Japanese language 'Learn Sangaku' *Sangaku ni manabu* 算額に学ぶ by Ohara Shigeru which also gives solutions to 95 problems in the same form. This indicates there is a gap in the field regarding the actual native methods of the Edo period for creating and solving *sangaku*.

Another area which has not been investigated is that of the role and function of diagrams in the Japanese tradition. In the last century much work has been done exploring the role of diagrams in the Greek tradition, and whether certain mathematical problems are underdetermined or overspecified. Diagrams are a key feature of *sangaku* tablets, but to date no investigation of the interaction between the diagram and text, and what they tell us about the Japanese tradition as a whole, has been conducted. As well as this, a detailed examination of whether *sangaku* problems should be considered Euclidean style theorems and proofs has not been produced.

To conclude, it can be seen that scholars agree that *sangaku* functioned as a mode of displaying mathematical results to a wider audience. However, there has become a wider set of opinions since 1990 regarding whether they also functioned as religious offerings and art. In recent times opinion seems to be moving back towards considering these tablets as more than promotions of mathematics, but no solid investigative work has been conducted to see if there is any evidence to support this, and, not all authors agree. As Heeffer writes in his 2012 article, still "Little is known about the purpose of the tablets and the intentions of the author. Recent publications on *sangaku* seem to avoid this important question" [28, p. 277], indicating the need for this issue of the function of *sangaku* to be readdressed and thoroughly investigated. Also, as Lu has indicated, the traditional methods of creating and solving *sangaku* are missing in the current literature, and in need of research. Additionally to this, how *sangaku* fit within the wider mathematical tradition of the period and the role of their diagrams is unclear.

1.2 The Nature of Sangaku

As shown in the brief literature review above, there is disagreement regarding whether *sangaku* were intended as religious offerings, works of art, or modes of mathematical communication. Also, there is no investigation into whether *sangaku* did function as proofs and theorems, how these tablets were originally created and solved using traditional methods, how they fit within the broader mathematical community of the time, and the role their diagrams played.

I promote a position in which *sangaku* should be considered to have multiple dimensions as mathematical, social, cultural, and religious artifacts. Through an examination of the tradition and some example tablets, I show that *sangaku* can encompass all the qualities and functions mentioned in the literature, and are not exclusively challenges and objects of communication nor exclusively works of art and religion. As well as arguing for these positions, I attempt to fill the gap in the literature illustrated by Lu by presenting some traditional Japanese methods side by side with modern mathematical methods to show how *sangaku* could have been solved traditionally, the style they may have used, and how they connected with the wider mathematical tradition through their content and methods. I also investigate the role that diagrams played, showing that when abstract geometrical problems are presented diagrams are vital for determination - but, in turn - diagrams vitally depend on their accompanying text.

In presenting my arguments, I first introduce and outline the history, general format, style, authors, and nature of the *sangaku* tradition. Some unsolved *sangaku* are also solved using modern methods to illustrate the content of the tablets and show what a modern analysis of these problems looks like. These problems form part of the primary source material from which the conclusions of this thesis are drawn.

In chapter 3, I present my argument that *sangaku* did function as communication devices, were part of the broader mathematical tradition of the Edo period, and classified as examples of conventional geometry problems. This is evidenced by examining some examples of mathematics of the Edo period alongside *sangaku* problems. These problems are presented in their original form, with an accompanying transliteration and English translation to illustrate how actual Edo mathematics presented; helping to alleviate the gap in the literature exposed by Lu. These traditional methods are applied to *sangaku* and used to support the argument by scholars such as Horiuchi that *sangaku* did act as transmission devices for communicating standard problems in geometry, and examples how problems could be solved using actual methods available at the time.

Later in chapter 4, I show how *sangaku* constitute examples of *ōema* offering tablets, and functioned for various audiences as religious and artistic works, supporting the argument by authors such as Rothman and Heeffer that they were acts of worship and works of art.

In chapter 5, the role of diagrams in the Japanese tradition is examined. I show how *sangaku* problems crucially rely on diagrams, and are underdetermined without them. However, there is also a necessary interdependency of the text and diagrams, with both being vital when abstract geometrical problems are presented. It is also seen how diagrams had pedigogical and ornamental functions through an examination of their colours and styles.

1.3 Original Sources

Many original sources were used in the production of this research. A number of *sangaku* included and researched were photographed within Japan by myself in the summer of 2012 and in March-April 2014. Other tablets included in this research have been sourced from the `http://www.wasan.jp` website run by Dr Hiroshi Kotera, which is the primary resource for *sangaku* researchers outside of Japan. The site includes many photographs of *sangaku* tablets. However while the website provides excellent examples of *sangaku*, many of the photographs require updating to a higher quality. Dr Kotera also supplies an extensive online archive of digital *wasan* texts from the Edo and Meiji periods at `http://www.wasan.earth.linkclub.com/archive.html`. As well as this, a variety of Edo and Meiji period books have been scanned and put online by Waseda

University in Tokyo. However, at the time of this project many of the texts have become unavailable. They can still be accessed in person at the Waseda Library upon registering and making a request.

The main original mathematical texts of the Edo period I have relied on for this research are the *Jinkoki* 塵劫記, *Sanpo Tenzan Shinan* 算法点竄指南, *Sanpo Tensei Shinan* 算法天生法指南, *Sanpō Shinshō* 算法新書, *Sanpō Jojutsu* 算法助術, and *Sanpo Jikata Taisei* 算法地方大成.

The *Jinkoki* 塵劫記 was first published in 1627 by Yoshida Mitsuyoshi. There were subsequent publications up to November 1641, and numerous illict versions were published during the Edo period. It is often considered the best selling book of the Edo period due to the number of copies created and circulated. The copy I have referred to is an English translation by the Wasan Institute from 2000 (see [55]).

The *Sanpo Tensei Shinan* 算法点竄指南 by Aida Yasuaki, published in 1810, lists a collection of geometry problems, some which are taken from the *Sanpo Shinpeki* 神壁算法 by Fujita Sadasuke which records *sangaku* problems. It also shows how to solve various problems using the *tenzan jutsu* technique, though a slightly different presentation of the technique is used. An online version of this text can aso be found on Dr Kotera's website (http://www.wasan.earth.linkclub.com/archive.html).

The *Sanpō Jojutsu* 算法助術 is a text containing formulas of Japanese geometry. It was originally published in 1842 by Yasunoshin Yamamoto. The original Japanese text can be found on Dr Kotera's website (http://www.wasan.earth.linkclub.com/jojutu/jojutu.html) as well as an unpublished English translation by the Chairman of the Nagano prefecture Wasan society, Nobuya Nakamura (http://www.wasan.earth.linkclub.com/kosiki/kosiki.html).

The *Sanpō Shinshō* 算法新書 was original published in 1830. I have worked from an original fourth edition paperback copy from 1880 bought from an antiques dealer. The *Sanpō Shinshō* is a comprehensive mathematical textbook that contains instructions on using the Japanese rod and bead abacuses as well as the *tenzan jutsu* algebra method. It also provides examples of how to use general geometrical principles to solve specific problems given in the *sangaku* style.

The *Sanpo Jikata Taisei* 算法地方大成 from 1838 is a five volume work dedicated to teaching surverying using a compass, protractor, and measuring square. It was written by Hiroshi Hasegawa and Hodo Akita. A version of this text bought from an antiques dealer was used in this thesis.

The *Sanpo Tenzan Shinan* 算法天生法指南 was written by Ohara Toshiaki in 1810. It introduces the reader to a variety of problems on geometrical topics, which are then solved using the *tenzan jutsu* symbolic manipulation technique. A copy of the original text can be found on Dr Kotera's *wasan* website (http://www.wasan.earth.linkclub.com/tenzan/tenzan.html) as well as in the Waseda University online library of rare books.

1.4 A Brief History of Mathematics in Japan

Mathematics as a discipline first began to develop in Japan during the era of the Kinmei emperor (510-570 CE). It was sparked by the arrival of an expert on the Chinese calender from China, and soon after a large influx of Chinese learning began to flow into Japan via Korea [65, p. 1]. While Chinese learning spread throughout Japan, it was not until the early eighth century with the start of the Nara period (710-794) that centres of mathematics began to be established with the intention of training government officials [65, p. 1-2]. The first university system itself established by Emperor Mommu (697-707) in 701 had nine Chinese mathematical texts in the curriculum [87, p. 9]. The Nara period was an era of cultural, political, and economic development. It saw the Japanese embark on the great feat of building a new capital city on the Kansai plains - now modern day Nara city. The creation of mathematical departments and increased study of Chinese mathematics was most likely due to the need for calculation in areas such as taxation, architectural planning, civil engineering, astronomy and the calender which were encouraged by the social and political environment [65, p. 2]. But though mathematical study did increase, it was still mainly used for practical matters and remained at this level until the Edo period (1603-1868).

The Edo period began after the first of the Tokugawa *shōguns* – Ieyasu Tokugawa 徳川家康 – rose to power and brought unity to the country in 1603 after years of civil war [55, p.14]. The new *shōgunate* feudal regime instigated many new changes and policies, the most important and influential being the shifting of the capital from Kyoto to Edo (modern day Tokyo) and the national seclusion policy *sakoku* 鎖国. As a response to the rapid growth of Christianity within Japan at the time, in 1612 Ieyasu issued a proclamation which ordered the extermination of all Catholics [77, p. 290]. In 1630, the third Tokugawa *shōgun* Iemitsu 徳川家光 (1604-1651), went further with another proclamation that prohibited importing and selling European and Chinese books dealing with Christianity [77, p. 290] Later, in 1639, he enforced the national seclusion policy which saw all foreigners banned from Japan other than select Chinese and Dutch traders at Nagasaki port. The ban on importing foreign works on Christianity was known as the Edict of Kanei, and censorship of foreign books continued under this law until 1720. The edict had a significant impact on mathematics, as it saw books banned in two areas, the first being religion and the second science [77, p. 292]. On one particular list of banned books uncovered, only seven texts of twenty involve Christianity, with the rest being scientific works on mathematics, astronomy, and geography [77, p. 294]. Because of this ban on Western as well as Chinese texts, it was difficult for citizens to access and study mathematics during this time.

It was in this environment that native Japanese mathematics known as *wasan* 和算 (where *wa* 和 is a term referring to Japan and *san* 算 is calculation) came to life and flourished. *Wasan*, while originally based upon Chinese mathematics, quickly branched off and evolved into something uniquely Japanese. The most significantly different *wasan* practice was the creation and dedication of *sangaku* mathematical tablets in shrines and temples. There is no parallel practice of mathematical tablet creation or dedication in other parts of Asia. Because of this, *sangaku* are excellent primary sources of distinctly Japanese mathematical practice, and their study is a vital for understanding *wasan*.

While the ban on foreign scientific works was eased in 1720 by Yoshimune Tokugawa 徳川吉宗, *shōgun* Yoshimune only made these works available to certain officially recognized scholars in his office, meaning mathematicians outside of the *shōgunate* were still unable to access the works for some time. When books did manage to trickle down over the next century, they failed to have an immediate impact on mathematicians. Koizumi explains that:

> The *wasan* mathematicians claimed that in mathematics Japan was better than the West. They were proud of having developed a mathematics unique to Japan and of having created only the pure, as opposed to applied, science among the traditional sciences [43, p. 12].

It was not until 1857 that a text on Western mathematics using Western notation and computation was published, meaning *wasan* and native methods were used for the majority of the isolation period [43, p. 12]. Then in 1853, the American Commodore Matthew Perry came to Japan and demanded the country open ports for whaling ships and establish trade [43, p. 5]. Thirteen years later in 1868, Japan abandoned its rigid seclusion policy, and a new government under the reign of Emperor Meiji was established [43, p. 6]. In 1877, what would later become Tokyo University was formed. In the Universities built in Japan in the following years Western mathematics was favoured over *wasan*, and from this period on the native tradition began to decline. Although some practitioners continued creating *sangaku* into the early 1900s, the practice came to an almost complete stop with World War II. While *wasan* is no longer in practice, having been fully supplanted by Western mathematics, the creation of *sangaku* tablets using old and modern methods has seen a revival since the 1970s, and many new tablets continue to be dedicated each year.

1.5 Translation Methodology and Notes

The translation of original sources is an intensive process which brings a unique set of issues and concerns on how to best represent the original text. As many *sangaku* and *wasan* texts in this thesis have been translated from original sources, these concerns apply here also.

A variety of different translation approaches have been used by historians working in the fields of Babylonian and Greek mathematics. In this section, I discuss some of these methods, as well as those currently used in the field of Japanese History of Mathematics. I also explain the method of translation I have adopted for this work.

1.5.1 Translation in the History of Mathematics

There have been a few different approaches historically to the translation of mathematical texts. Among the camps are those in favour of literal translations, and those who feel functional translations that appeal to a modern audience are more appropriate.

Literal translations are those which aim to express mathematical texts in a way that stays as true to the original text as possible. As Mathieu Ossendrijver discusses in *Babylonian Mathematical Astronomy: Procedure Texts*, in such approaches "word order is usually maintained, even at the cost of English grammar, and each...word is always translated by the same English word" [67, p. 13]. This approach tends to avoid the imposing of modern mathematical language and terms in the translation, as these were not available to the original authors. For example, Unguru, an advocate of translations without recourse to modern mathematical concepts, writes

> To read ancient mathematics texts with modern mathematics in mind is the safest method for misunderstanding the character of ancient mathematics⋯..To assume that one can apply automatically and indiscriminately to any mathematical content the modern manipulative techniques of algebraic symbols is the surest way to fail to understand the inherent differences built into the mathematics of different eras [96, p 86].

Unguru's approach aims for translations to be as close and faithful as possible to the original text by having the translator themselves get into the mindset of the mathematician who wrote the text. This involves keeping in mind and using the tools and methods available to the author. However, for historians to first know what these tools and methods were, they must translate texts. But since translations should only be done using concepts known at the time a troubling circle presents, meaning it is important to have some starting point where modern methods are used to inform translation.

An alternative approach is to produce translations which function in the same way as the original text but are presented in a manner which readers can easily comprehend. This is referred to as the functional method by historians such as Ossendrijver [67, p. 14]. While this allows the modern reader to better understand the translation, there is a risk that the translator may unconsciously embellish the text or translate the mathematics using terms and concepts not available to the original author(s).

1.5.2 Translating Wasan: Current Approaches and Problematics

In the current literature on *sangaku*, the majority of translations are functional rather than literal, focusing on presenting the problems with modern language and mathematical terminology. Many translations provide additional grammar and language to make the original text easier for the reader to understand. Some even include modern mathematical characters and formulas. English translations of *sangaku* are rare, and the majority are translated into modern Japanese. In this section I examine the translation style of two different authors who have published work on *sangaku*. I take one example from each author and present it along with 1) a transcription of the original text, 2) a literal translation in English of the original text, and where necessary 3) a translation in English of the Japanese translation given by the authors.

1.5.2.1 Example 1: Kijimadaira Tenman-gū Problem

The first example of a *sangaku* translation which I will examine comes from the Kijimadaira Tenman-gū *sangaku*. This has been translated into modern Japanese by the Nagano Wasan Research Society in the text 木島平村の和算 *Sangaku of Kijimadaira.*

The transcription and a literal translation for the tablet are as follows:

Transcription	**Literal Translation**
今如圖直線五圓載有只云 甲圓径二寸乙圓径三寸 丁圓径幾	There is diagram line five circle on say *ko* diameter 2 *sun otsu* circle diameter circle 3 *sun tei* circle diameter how much.
答日丁圓拾八寸	Answer *tei* circle 18 *sun*
術日以甲圓径二段與乙圓径差 乙圓径除自之甲圓径乘問合 得乙圓径合問	Technique by means of *ko* circle diameter 2 and *otsu* circle diameter subtract *otsu* circle diameter divide square *ko* circle multiply obtain.

Nagano Wasan Society Translation

The translation in modern Japanese given by the Nagano Wasan Society is provided below

> 甲, 乙, 丁円は, 隣の円と外接し, 直線に接している。また, 丙円は他の 4 円に外接している。甲, 乙円の直径がそれぞれ 2 寸, 3 寸のとき, 丁円の直径を求めよ。
> [答] 丁円径 18 寸
> [術] 甲, 乙, 丙, 丁円の直径がそれぞれ d_1, d_2, d_3, d_4 とすると,
> $d_4 = \dfrac{(d_1 \cdot d_2)^2}{(2d_1 - d_2)^2}$ [89, p. 43].

Translated into English, this reads:

> The circles *ko*, *otsu*, and *tei* are neighbouring and circumscribe the line. Also, four other circles circumscribe circle *hei*. When the circles *ko* and *otsu* are 2 *sun* and 3 *sun* respectively, find the diameter of circle *tei*.
>
> Circle *tei* diameter 18 sun
>
> Using circles *ko*, *otsu*, *hei*, and *tei* respectively as d_1, d_2, d_3, and d_4.
> $d_4 = \dfrac{(d_1 \cdot d_2)^2}{(2d_1 - d_2)^2}$

Comments

The translation of the Nagano Wasan Research Society is not a literal translation, as it includes terminology not present in the original text in order to be more palatable. For instance, the text uses the term *gaisetsu* 外接, which is translated in the *Mathematics English-Japanese and Japanese-English Dictionary* as "circumscription" [105, p. 190]. The original text does not contain this term, or reference the four circles circumscribing the circle *hei*. Instead we are only told that there are 5 circles on a line, and given the diameters for three of them.

The technique section provided by the Nagano Wasan Research Society also differs considerably from the original text. Instead of a translation of the original technique section, completely different text is included which assigns the modern variables d_1, d_2, d_3, and d_4 to the circles and presents the original technique section in terms of an algebraic formula. These additions of extra terminology and modern mathematics cause the translation to differ considerably from the original text presented on the tablet. The creators of *sangaku*, though having knowledge of symbolic manipulation through *tenzan jutsu* (see Chapter 3 for more details on this), specifically chose to present solutions verbally in the technique section. By replacing this text with modern algebraic formulas, the text no longer accurately represents the original problem and intention of the author.

1.5.2.2 Example 2: Konnou Shrine Problem

A rare instance of a *sangaku* problem translated directly into English by a Japanese scholar is the Konnou Shrine tablet problem. This translation was done by Hideyo Makishita. The transcription and a literal translation for the tablet problem are as follows:

Transcription	**Literal Translation**
如圖中圓径九寸小圓径 四寸大圓径幾何問	There is diagram medium circle diameter 9 *sun* small circle diameter 4 *sun* big circle diameter how much problem.
答三十六寸	Answer 36 *sun*
術日置中圓径除小圓径 開平方内減一個目之以除 中圓径得大圓径合問	Put medium circle diameter divide small circle diameter square root inside subtract 1 *ko* square by means of medium circle diameter obtain big circle diameter required.

Hideyo Makishita's Translation

The English translation of Makishita is as follows:

> As shown in the illustration, if the diameter of the medium circle is 9 *sun* and the diameter of the small circle is 4 *sun*, what is the diameter of the large circle?
>
> Answer 35 *sun*
>
> Explanation (formula): first divide the segment of the medium circle by that of the small circle and take the square root of that number. Then, subtract 1 from that number and square the result. By dividing that number by the segment of the medium circle, we can find the segment of the large circle. The resulting number is equal to the diameter of the large circle [50, p. 140].

Comments

The translation given by Makishita is more functional than literal, and fares better than that produced by the Nagano Wasan Society. However, here too we find the technique section embellished with additional terminology not found in the original text. For instance the term ‘segment’ is used three times by Makishita, but the Japanese equivalent of this term is not found in the original text. The ending statement “The resulting number is equal to the diameter of the large circle” is also different to what is presented in the original text, which literally translates to ‘obtain big circle diameter required’. Makishita does however keep the original verbal presentation of the solution in the technique section, making his translation a much more accurate representation.

1.5.3 Translation Methodology Adopted

As shown in the previous examples, caution must be taken when translating *sangaku* in order to ensure the problems are accurately presented to a modern audience. In this section, I discuss the translation methodology I have adopted for my translations of *sangaku*.

The method I adopt is a mix of the literal approach of Unguru and the functional style of Makishita. Though I present the text in a way that is easier to follow for the modern reader - rearranging the word order to be more natural for the English reader for instance - I only present the original concepts expressed in the text. I leave in the original Japanese characters for labelling circles - *kō* 甲, *otsu* 乙, *hei* 丙, *tei* 丁, etc. The reason for this inclusion is that while this series can be translated to mean something similar to ‘first, second, third, fourth, fifth, etc.’, this language can cause confusion when these characters are used to label many circles in a diagram. I feel their inclusion does not impact the readability of the problems, and may in fact be useful since they are commonly repeated on *sangaku*.

This thesis uses a variety of Japanese terms in *kanbun* and modern Japanese. To aid readers with varying levels of Japanese ability, romanisation of Japanese in English is included where possible, as well as original Japanese text. However due to the difficulty with variation in language pronunciation, which has changed over time, it has not been possible to always provide the original pronunciation of *kanbun*. I use a combination of Japanese *on yomi* and *kun yomi* readings based on what readings have been available. *On yomi* is the traditional Chinese reading of Japanese characters, while *kun yomi* is native Japanese readings.

1.5.3.1 Technical Analysis

In order to accurately represent the nature of the technique section of *sangaku*, in chapter 2 a technical analysis section is included along with a modern analysis. The technical analysis section breaks down the technique section into the different actions that are being performed. Its purpose is to present the technique section in a manner that maintains the features of the original text, and gives as literal a representation as possible while showing how we interpret this in the context of modern mathematics.

Upon breaking down the technique section of a *sangaku*, the sense of action becomes apparent. For example, consider the technique section of the Yoshifuji Mishima *sangaku* examined in section 2.8.8:

> Technique: Put the short side squared and add the long side squared. Take the square root. Add the long side and name this heaven. Inside subtract the short side and double. Furthermore name this earth. Put 3 ko and take the square root. Multiply heaven and divide by the short side. Add 3 ko and divide into earth. Obtain the triangle side length as required.

This can be literally broken down into the following steps:

1. Put down the short side squared and add the long side squared and take the square root
2. Add the long side to (1) and name heaven
3. Subtract the short side from (2)
4. Double (3) and name earth
5. Put down 3 ko and take the square root
6. Multiply by (2)
7. Divide (6) by the short side
8. Add 3 ko to (7)
9. Divide (4) by (8)
10. Obtain the triangle side length

In terms of modern mathematics, this can be represented as the formula:

$$\frac{\mathit{Earth}}{\dfrac{\sqrt{3}\cdot \mathit{Heaven}}{\mathit{Short\ Side}}+3}$$

As seen in the previous section, sometimes in the literature only a formula such as above will be given for the technique section (e.g. the translation by the Nagano

Wasan Society). However, preserving the sense of action shown within the original text as in steps 1 - 10 above is important when examining *sangaku*. As mentioned, Japanese mathematicians had the ability to represent a formula symbolically using the *tenzan jutsu* symbolic manipulation method which is described in chapter 3. However, instead of a symbolic representation we are given a series of instructions. It is also the case that the language itself used within the technique section suggests it should be carried out as an action. For example, a common beginning to the technique section is 置 which in modern Japanese is the root of the verb *oku* 置く 'to put'. The common taking of the square root 方開之 also literally translates to "opening the square" [103, pp. 23], which uses the verb 'open'. As well as this there are instances where the character 名 appears indicating a series of calculations is to be 'named' by a certain term.

Because Edo period mathematicians did not use formulas, and rather provided instructions with active language, by only expressing the technique section as a formula we lose the original intended presentation. In trying to maintain and capture this sense of action, in the technical analysis section I provide two different renderings of the text:

1. Rendering 1 – Procedure to capture the literal and original presentation and how it would have been read and approached by the Edo period Japanese.
2. Rendering 2 – Formula to show how modern mathematicians can interpret this within the context of modern notation and presentation.

The technical analysis section is further described in section 2.8.1.

1.5.3.2 Modern Analysis

Along with the technical analysis, a modern mathematical analysis is also provided. This section shows how a *sangaku* can be solved using modern mathematical methods and language. The aim of this analysis is to illustrate to the reader just how much effort and calculation was involved in solving *sangaku* problems. While the set of actions producing the formula in the technique section can seem basic, the actual steps involved in obtaining that formula are strenuous. This section shows the work involved on the part of the creator of the problem and the person who attempts to solve it. We see the true nature of the technique section – deceptively simple. The calculations in this section are also designed to help the reader understand what these problems may look like when working is included.

1.5.3.3 Character Identification

When conducting this research, no dictionaries or guides were available for reading *sangaku*, which are written in the classical *kanbun* 漢文 language. The only text in English on reading *kanbun* in general is *An Introduction to Japanese Kanbun* by Akira Komai and Thomas H. Rohlich. In Japanese I have referenced *Kanbunpou kiso honto ni wakaru kanbun nyumon* 漢文法基礎本当にわかる漢文入門 by Kaji Nobuyuki and *Kanbun kihongo jiten* 漢文基本語辞典 by Shigeyuki Amano. I have relied on a combination of these dictionaries as well as modern Japanese and Chinese translations to translate and understand the language of *sangaku*.

1	一	丨	丶	丿	乙	亅	2	二	亠	人	儿	入	八	冂	冖
冫	几	凵	刀	力	勹	匕	匚	匸	十	卜	卩	厂	厶	又	3
口	囗	土	士	夂	夊	夕	大	女	子	宀	寸	小	尢	尸	屮
山	巛	工	己	巾	干	幺	广	廴	廾	弋	弓	彐	彡	彳	4
心	戈	戸	手	支	攵	文	斗	斤	方	无	日	曰	月	木	欠
止	歹	殳	毋	比	毛	氏	气	水	火	爪	父	爻	爿	片	牙
牛	犬	5	玄	王	瓜	瓦	甘	生	用	田	疋	疒	癶	白	皮
皿	目	矛	矢	石	示	禸	禾	穴	立	6	竹	米	糸	缶	罒
羊	羽	耂	而	耒	耳	聿	肉	臣	自	至	臼	舌	舛	舟	艮
色	艹	虍	虫	血	行	衣	西	7	見	角	言	谷	豆	豕	豸
貝	赤	走	足	身	車	辛	辰	辶	阝	酉	釆	里	8	金	長
門	阝	隶	隹	雨	青	非	面	9	革	韋	韭	音	頁	風	飛
食	首	香	10	馬	骨	高	髟	鬥	鬯	鬲	鬼	11	魚	鳥	鹵
鹿	麦	麻	12	黄	黍	黒	黹	13	黽	鼎	鼓	鼠	14	鼻	齊
15	歯	竜	亀	龠											

Table 1.1: The 214 Japanese Radicals

To identify characters on *sangaku* for use in transcriptions, I have used a method of identification from radicals. This involves the breaking down of a *kanbun* character into the different elements which make it up. For instance, the character 文 can be broken down into the two radicals 文 and 口. A table of the radicals of the Japanese language can be found in Table 1.1.

Chapter 2

The Sangaku Tradition

2.1 Introduction

During the Edo period, Japan found new unity and peace under the Tokugawa *shōgunate*, and the economy thrived creating a time of great prosperity. In this environment, culture and art reached new heights, and *sangaku* were born. The function of these tablets is still a matter of some debate. In this chapter, I will consider whether *sangaku* should be treated as forms of mathematical communication and promotion as well as religious and artistic works. The *sangaku* tradition will first be introduced, with a discussion of what these tablets look like, who was making them, and the type of problems they dealt with. I break down a typical *sangaku* and how it presents visually, before exploring the history of these tablets, their location, and their features. After this I examine who the mathematicians creating *sangaku* were, the type of mathematics they were doing, and how they were doing it. After this a selection of *sangaku* problems are examined to show the typical style and subject matter of the tablets. Transcriptions and translations of the problems are provided, and where possible, the problems are solved with modern mathematical methods. This modern approach is later contrasted with solutions involving traditional Japanese methods in chapter 3.

2.2 Dissection of a Sangaku

Sangaku, as previously stated, were wooden mathematical tablets found in Shinto shrines and Buddhist temples. They were highly individualistic in nature, with each tablet being unique in size, theme, and design. They also had a fixed location in a different, specific religious site. *Sangaku* appeared throughout Japan in Edo times, being dedicated to sites as grand as the Gion shrine of Kyoto, or smaller, rural places of community worship. While there is great variation in the *sangaku* tradition, the majority of tablets are presented using a similar format.

A *sangaku* can usually be broken up into six sections. I use as an example the Chosekiji shrine in Iida, shown in Figure 2.1, which displays how a typical *sangaku* can be presented and read. In my examination of specific *sangaku* later in this chapter, I focus attention on three of these sections - the problem, answer, and technique. Chapter 5 however is dedicated to examining section one on diagrams. In cases where it is pertinent attention is given to other sections. My focus on the four sections of diagram, problem, answer, and technique is due to these being the most straightforward elements on the tablets to translate and investigate.

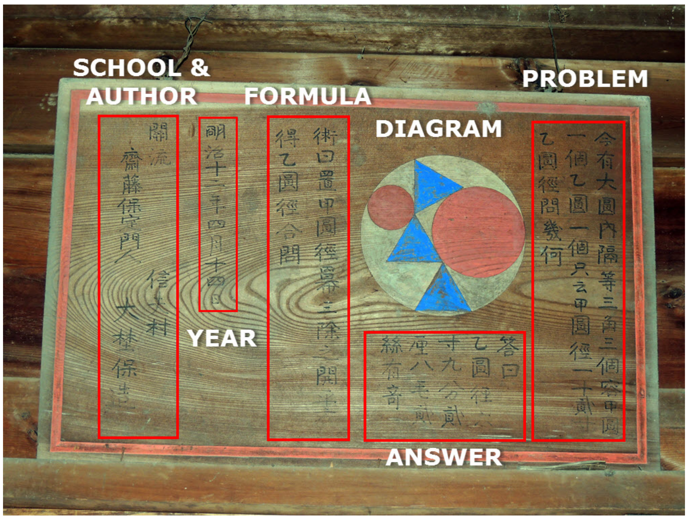


FIGURE 2.1: Sections of the Chosekiji *sangaku*. (Image by author).

Transcription	**Translation**
今有大圓内隔等三角三個 容甲圓弐寸一個乙圓一個 只云甲圓径一十乙圓径問幾何	There is a large circle (大) which contains three equal sized equilateral triangles. Contained inside is one circle *kō* 甲 and one circle *otsu* 乙 . Say the diameter of *kō* 甲 is 12 *sun*. Problem - what is the diameter of circle *otsu* 乙?
答日乙圓径六寸九分弐厘 八毛弐絲有奇	Answer: The diameter of *otsu* 乙 is 6 *sun* 9 *bu* 2 *rin* 8 *mo* 2 . . .
術日　置甲圓径零三除之 開平方得乙圓径合問	Technique: Put the diameter of circle *kō* 甲 squared. Divide by three. Take the square root. Obtain the diameter of *otsu* 乙 as required.

TABLE 2.1: Right: Transcription of Chosekiji *sangaku* text. Right: English Translation

1. **Diagram**

   The most visible element of a *sangaku* is the diagram. This usually is made up of brightly coloured geometric shapes and located towards the top of the tablet. The diagram usually takes up a large proportion of the tablet, presumably being intended to catch the attention of the observer. In most cases, the size of the diagram does not match the dimensions given to describe it in the problem text. There are also examples where the shapes contained within the diagram are not drawn with correct proportions. Diagrams are further discussed in chapter 5.

2. **Problem**

   This section usually begins with the characters *kenrijozu* 今有如圖, which form the almost universal introduction to *sangaku*. Kazuyoshi Wakuta and Kazuhito Togawa translate this into the modern Japanese *zu no yō ni* "図のように" [98, p.18]. In English this reads "as in the diagram" or "as depicted/drawn". It refers the observer to the geometrical figure depicted on the tablet. However, some tablets such as the Chosekiji do not include *jozu* 如圖, and can be interpreted as reading "There is...".

   Following this, the standard presentation of *sangaku* is to next describe what the observer sees in the diagram. They are told what shapes are present, and how many of each there are. On the Chosekiji tablet, the text describes one large circle containing three equal equilateral triangles. There are also two circles labelled using characters from the Chinese calender (detailed further in section 2.8.5).

   Once the diagram has been described, if there are numerical values relating to any of the given figures usually next the author indicates what they are. For the Chosekiji tablet, the bigger of the two inner circles is given as 12 *sun*[1]. When values are to be given, usually the characters *tadashi* 只云 or *tadaiu* 只言 appear first, which translate to "Say..." in the sense of "suppose".

   The final part of the problem section is the problem itself. This begins with the character *mon* 問 - which translates to "Problem" - and ends with *ikubaku* 幾何 - which translates to "what is" or "how much [is]". This section characteristically asks for the value of one of the figures in the diagram. It is usually structured to ask "What is the value of [figure]?".

3. **Answer**

   This section begins with the heading *kotae-ni-iwaku* 答曰, which can be translated into modern Japanese as *kotae-ni-iwaku* 答 meansing 'response', 'reply', or 'answer'

[1] *Sun* is a measurement unit used in Japan. It is roughly equivalent to 3.03cm. See section 2.3 for more details on measurement units in Japan.

[98, p. 19]. This can be thought of as the answer section. It always directly follows the problem section and presents in one of two forms. Firstly it may be a numerical value for the sought after figure. Secondly, it may instead use the characters *go hidari jutsu* 合左術 "See the technique on the left", or some variation. This directs the reader towards the technique section for the answer since the formula does not produce a numerical value.

When a numerical value is given, it is typically expressed using the traditional *sangaku* counting system (see section 2.3). When a value has many significant figures, the characters *yuki* 有奇 are placed after the values to indicate that not all figures have been included. I have expressed this using a series of dots - "..." - in my *sangaku* translations.

4. **Technique**

   The characters *jutsu-ni-iwaku* 術日 act as a heading for the technique or formula section. It is translated as *jutsu* 術 by Wakuta and Togawa [98, p. 19]. *Jutsu* 術 can mean 'art', 'way', 'method', or 'technique'. While *jutsu-ni-iwaku* 術日 labels the formula, I use the translation 'technique' rather than 'formula' because the character most commonly used to represent the term 'formula' *shiki* 式 is not used to label this section. The section reads like a set of instructions or recipe, which form a formula. The common operations that occur are multiplication, division, the taking of square roots, addition, subtraction, and squaring. The end of the formula is marked by the character *toku* 得 followed by the sought after figure and the characters *gōtoi* 合問. This combination can be translated as "Obtain [figure] as required".

   The purpose of this section is to give a procedure which can be applied to the problem to find the solution. While a large amount of calculation goes into producing the formula, the author rarely provides their working.

5. **Year**

   Sangaku are dated using the traditional Japanese dating system, which is an era calendar scheme. In this system years are labeled by combining an era name with a year number. This system is still in operation, with 2015 being year 27 of the Heisei 平成 era, which began in 1989 when Japan gained a new emperor.

6. **Author and School**

   Sangaku usually give the name of the author and the school they belonged to. There were many different schools, with the main being the Seki school founded by Seki Takakazu 関孝和 (? - 1708).

2.3 History

How a typical *sangaku* presents visually has been shown, but how did this tradition develop in Japan? Prior to the creation of *sangaku*, there was a long running "custom of hanging tablets at shrines...centuries before sangaku came into existence" [71, p. 85]. The tablets commonly hung in shrines were known as *ema* 絵馬 – where *e* 絵 means picture or painting and *ma* 馬 a horse. In ancient times, "shrines used to keep live horses for ceremonial purposes" [39, p. 116], for they were considered messengers of the Japanese gods. However due to the expense of regularly sacrificing horses "paintings executed on flat wooden surfaces...were created as a modern substitute" [39, p. 116].

FIGURE 2.2: A *sangaku* (middle bottom) amongst *ōema* at Kitano Tenman-gū Shrine, Kyoto. (Image by author).

In the modern day, *ema* have developed into small wooden tablets displaying pictures of various animals (generally of the zodiac) on one side and a blank area on the other to write one's wishes to the gods. They are hung in specific areas of shrines and temples, and burned by the acting priest when there is a sufficient number. During the Muromachi (1333 – 1573 CE) and Edo periods, *ema* evolved into free standing flat boards known as *ōema* 大絵馬 (where *ō* 大 means large). They became a kind of folk art which allowed people to express their wishes in a highly visual manner. They would be offered to a shrine or temple in furtherance of a request, or as an expression of gratitude [69, p. 30]. As their name suggests, they were large in size, commonly being over one metre in length and width [70, p. 47-8]. Painted by professionals commissioned by rich donors, their subjects included historical figures, ships, scenes of battle, and classical stories [70, p. 47-8]. Due to their more exalted status, they were kept indoors or hung in a specific gallery style area of temples and shrines [70, p. 47-8].

Ōema were located in Shinto shrines and Buddhist temples because these traditionally played a significant public role in Japanese society. Shinto shrines are intimately connected with indigenous Japanese deities. In order to seek good fortune from these deities, members of the community frequently worship at these shrines. As well as this, many ceremonies and rites of passage are conducted in these locations, such as birth ceremonies and weddings. Shrines and temples also have annual festivals which involve the whole community, and at special times of the year visits to these locations are common and expected (such as New Years). Shinto shrines and Buddhist temples thus have traditionally functioned as both community locations as well as religious ones. Because of this, they were the perfect place to promote oneself as an artist or give an offering through the hanging of an *ōema*.

This tradition also provided mathematicians with a means to display their work at a time when mathematics was flourishing, so soon *sangaku* began to appear alongside *ōema* in shrines and temples. The *sangaku* tradition built upon that of the *ōema*, which provided the perfect template and space for displaying mathematics. However, whether *sangaku* had the same function as *ōema* is a different matter. This is discussed in chapter 4, where I argue that *sangaku* should be treated as *ōema* as well as forms of mathematical communication and promotion.

While it is estimated that there are only around 900 *sangaku* in existence today, records indicate that during the Edo period thousands more were produced [29, p. 9]. The oldest surviving tablet dates back to 1683, though the Edo mathematician Yamaguchi Kanzan references an older work created in 1668 [29, p. 9]. In the literature there is no mention of tablets created before 1600, most likely caused by a lack of accessible material on mathematical topics and the heavy focus on studying and reiterating Chinese learning. The great loss of tablets seen over time can be put down to numerous natural disasters which have occurred in Japan (for instance, the latest 2011 earthquake and tsunami are believed to have destroyed tablets in Fukushima and Iwate prefectures), the impact of wars, and the general decay of the tablets.

The tradition of creating and dedicating *sangaku* continued into the Meiji and Taishō (1912-26) periods, but came to an almost complete stop by the early Shōwa period (1926-89) and beginning of World War II. A revival of the tradition occurred during the late Shōwa period and has continued during the current Heisei (1989-) period.

Prefecture	**Number of Tablets**
Hokkaido	0
Aomori	3
Iwate	102
Miyagi	49
Akita	8
Yamagata	37
Fukushima	111
Ibaraki	21
Tochigi	21
Gunma	78
Saitama	88
Chiba	33
Tōkyō	16
Kanagawa	6
Yamanashi	5
Niigata	27
Toyama	17
Ishikawa	16
Fukui	23
Nagano	54
Shizuoka	7
Aichi	16
Gifu	8
Mie	14
Shiga	11
Kyōto	17
Ōsaka	13
Hyōgo	27
Nara	5
Wakayama	1
Tottori	0
Shimane	0
Okayama	25
Hiroshima	4
Yamaguchi	0
Tokushima	0
Kagawa	7
Ehime	31
Kōchi	0
Fukuoka	8
Saga	1
Nagasaki	3
Kumamoto	0
Ōita	1
Miyazaki	0
Kagoshima	0
Okinawa	0

Table 2.2: Number of Survived Sangaku per Prefecture [22, pp. 138-139]

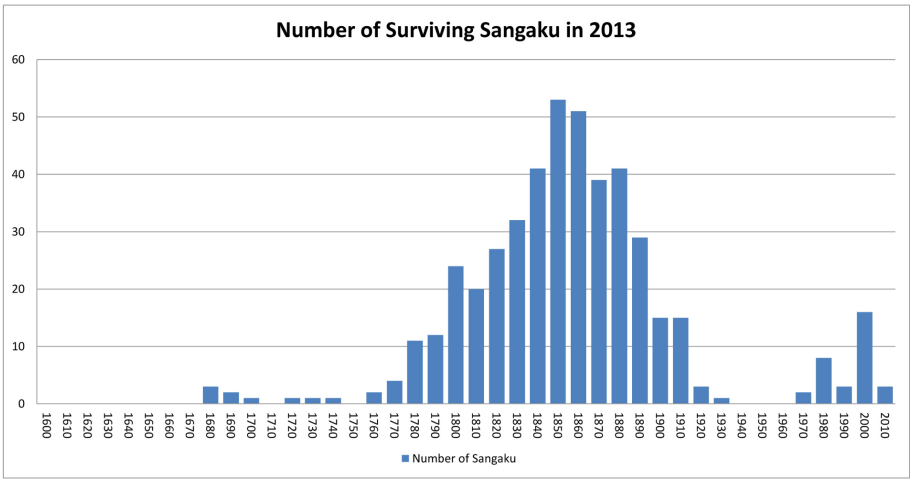


FIGURE 2.3: Estimated number of surviving *sangaku*[2].

Figure 2.3 displays the estimated number of currently existing *sangaku* as recorded at `http://www.wasan.jp` per 10 year block from 1600 to 2010. This data does not represent all *sangaku* created however, for only surviving tablets are included. There may be a higher percentage of later Edo period tablets recorded, which are more likely to still be preserved. However, from the data it appears the majority of tablets on the website were created between 1770 and 1900, with the peak years being 1850 and 1860.

In Table 2.2, the number of tablets per prefecture is listed. From this graph it can be seen that the greatest number of surviving *sangaku* are in Nagano prefecture, with Ehime and Iwate having the next highest numbers. Based on a report by the Statistic Survey Department of the Japanese Ministry of Internal Affairs and Communications, in 1920 Tokyo, Osaka, Hokkaido, and Fukuoka were the most populated prefectures [90]. Nagano, Ehime, and Iwate prefectures only ranked as the seventh, twenty-second, and twenty-ninth most populated for the period. The reason for the lower number of tablets in higher populated regions is currently unexplored, but may be due to the heavier destruction in larger metropolitan regions during World War II.

The highest number of *sangaku* are in Fukushima prefecture, with Iwate and Saitama having the next highest numbers. This likely because these more rural prefectures “did not at once provide Western-style calculation courses, but continued to print and disseminate [their] own text materials and teacher’s guides” [72, p. 214] when the country concluded the isolation policy in the mid 1800s.

[2]Reference [22, pp. 138-139].
[3]Public domain image courtesy of Wikimedia commons.

FIGURE 2.4: Prefectures in Japan[3]

2.4 Location

Sangaku most commonly appear in an area of shrines known as the *emaden* or *emadō* 絵馬堂, built to store and display *ōema* [69, p. 30]. One of the oldest in Japan is the Kitano Tenman-gū *Emadokoro* 北野天満宮絵馬所 (Figure 2.5) – or "ema place" – of the Kitano Tenman-gū shrine which dates from 1606-1608 [69, p. 30]. A *sangaku* is found in this particular *emadō* amongst a variety of *ōema*, as seen in Figure 2.2. These halls functioned as informal public art galleries where budding artists could display their skills [69, p. 30].

Sangaku also appeared on the eaves of the main shrine and temple halls. In some instances, they were also situated attached to a smaller shrine located within the larger complex.

FIGURE 2.5: *Emadō* containing a *sangaku* (not pictured) at Kitano Tenman-gū shrine, Kyoto. (Image by author).

Of the mentioned *sangaku* which appear attached to sub-shrines part of larger complexes, often the shrines they are part of are of the *Hachiman* 八幡神 or *Tenman-gū* 天満宮 variety. *Hachiman* shrines are the most popular in Japan, making up around half of all shrines. *Hachiman* is a Japanese god of war and protection, who is associated with the the ancient Emperor *Ōjin*. *Tenman-gū* shrines enshrine the vengeful spirit of the exiled scholar *Sugawara no Michizane* as the god *Tenjin* 天神. *Tenjin* is "venerated as the tutelary deity of scholarly and literary activities" [40, p. 148] amongst the Japanese. The most common visitors to *Tenman-gū* shrines are secondary and University students, seeking help with examinations. *Tenjin* is also the god connected with mathematics, and every January the *soroban hajiki zome* そろばん弾き初め - first practicing of the Japanese abacus for the new year - occurs at the Kitano Tenman-gū shrine in Kyoto where *Michizane* was originally enshrined. An example of a *sangaku* part of a *Tenman-gū* sub-shrine is that dedicated to the Ikutama shrine in Osaka (see Figure 2.6).

Figure 2.6: *Sangaku* above the *Tenman-gū* subshrine of Ikutama shrine, Osaka. (Image by author).

2.5 Authors, Methods, and Tools of Sangaku

The *sangaku* tradition was a unique mathematical practice of the Japanese Edo period. But who were the authors of *sangaku*, and how were these mathematical artefacts created?

The creators of *sangaku* came from all areas and classes of Japanese society, and it is believed that as "many as 8,000 mathematicians may have been active" [8, p. 20] in the Edo period. Originally, the majority of mathematicians were from the *samurai* class, and in 1650 up to "70 percent of mathematicians were samurai" [8, p. 19]. However these numbers steadily dropped, with just 35 percent being of the *samurai* class in 1850 and at the end of the period commoners from merchant and agricultural families making up "two-thirds of mathematicians" [8, p. 20]. This dramatic shift between the classes of mathematicians was the result of economic and political changes. In the Edo period "Merchants arose to become the new economic power" [65, p. 25] due to the economic boom that had occurred with the closing of the country. The politically motivated shifting of the capital by the Tokugawa shōgunate to Edo saw the "minor fishing village of little significance" [18, p. 23] grow to the extent that "by 1700 it had perhaps a million residents" [18, p. 23]. Its growth was the result of the alternative attendance policy which had forced the families of all *daimyō* 大名 (local lords) to permanently reside in the new capital and *daimyō* themselves to personally spend six months of every year there. Some also preemptively left family members there as a sign of good will.

The sudden and rapid increase in Edo's population brought on by these events had a dramatic impact on the local and national economy, as:

> ...[C]ommodities of every sort were funneled to the center...The provision of materials needed for life at the capital and transporting them there provided economic opportunities for commoners, and as the merchant and artisan classes grew in size and importance a new popular culture emerged [72, p. 128].

Due to these factors, Edo became "a centre of wholesale and retail trade on a grand scale" [79, p. 114]. The economy of Japan began to experience a period of rapid growth, and the merchant, farming, and artisan classes grew and blossomed in response to the increased need for goods and services occurring. The impact this had on the development of mathematics was profound. In order to "perform buying and selling, people needed to make calculations" [79, p. 114], making mathematics education a necessity. The creation of "non-technical books on mathematics for laymen became more and more popular" [65, p. 25], as more of these laymen needed to know mathematics, and their new found wealth allowed them to now engage in educational pursuits.

During the Edo period, many schools of mathematics also developed which acted similar to "merchant craftguilds" [65, p. 37], where mathematics was taught with a focus on secrecy. In these schools, "Teachers and pupils were organised in the form of the *iemoto* (家元) system" [64, p. 145]. This system was used for licensing, and is explained by Eiko Ikegami as follows

> In a typical large-scale iemoto, the grand master issued different levels of certification to students, while most of the actual instruction was carried out by local teachers. While the local teachers earned their incomes from tutoring students, the grand master received certification fees every time that student progressed to a higher degree [33, p. 165].

Those wanting to learn mathematics would join a school under this kind of system and work their way through the ranks by obtaining certifications in mathematics. While most mathematicians learned the secrets of the discipline through an *iemoto* system of mathematical teachers, the mathematician and lord Arima Yoriyuki (1724-1783) publicised many of the secrets of the school of Seki Takakazu [65, p. 37]. This made the techniques of mathematicians available to anyone who was able to read *kanbun*, the language many texts were written in. With the increasing education of the merchant, farming, and artisan classes, mathematical knowledge began to spread through all areas of society. Mathematicians in Japan, while being of different classes, were also of different genders and ages, as there are also examples of *sangaku* with problems created by young children of both genders.

While *sangaku* are beautiful visual displays of mathematics, their authors did not initially present their problems in this medium. A number of mathematical problems and works were first worked out and written down on paper, with some works then produced into manuscript form with woodblock prints and mass published later. During the Edo period, there were different types of paper used. High quality white paper was reserved for writing letters to those outside a family. For less formal situations, "flimsy, coarse sheets of grey, recycled paper" [102, p. 163] were used. This was due to the high cost of quality paper during the period. Although paper was expensive, the Japanese had an ample supply due to their eco-friendly recycling, with up to 100 percent said to have been recycled due to its ability to stand reuse [63, p. 8]. Mathematicians used low quality recycled paper rather than other mediums like sand for their rough working, and reserved higher quality white paper for their letters to other mathematicians, manuscript work, and outlines for *sangaku*.

As well as using paper for their rough working out of problems and manuscripts, mathematicians used the compass and carpenter's square for diagrams. According to legend, the carpenter's square and compass were invented by the first of the Chinese 'Three Sovereigns', Fu Xi who reigned during the mid-29th century BCE [13, p. 191]. While this is likely more myth than fact, paintings depicting these devices dating to the middle of the eight century have been discovered, indicating they were known and used in China from at least this period forward. The Japanese likely had access to these tools before the Edo period. Although the extent of their use in society is uncertain, it is known that both artists and mathematicians used them.

For instance, in *Hokusai's Geometry*, Christine M. E. Guth has researched geometry present in the works of the famous Edo period *ukiyo-e* artist Katsushika Hokusai 葛飾北斎 (1760- 1849). She has found that in his text *Ryakuga haya-oshie* 略画早押南 'Quick Guide to Painting' from 1812, Hokusai breaks down various drawings of objects into basic geometrical shapes such as circles and squares, and promotes the use of the compass and square for precise drawing [26, p. 127]. He writes

> If a mountain is ten feet...then a tree must be one foot, a horse an inch, and a man the size of a bean. So it is said about the laws [of proportions]. All things, however, originate in squares and circles. Here Old Hokusai will teach you how to become skilled in paintings of all kinds by using the compass and square...Having mastered these two instruments you can make precisely detailed (*saimitsu*) drawings on your own [26, p. 127].

Mathematicians also had access to these tools for constructing their *sangaku* problems. In the five volume text on land surveying *Sanpo Jikata Taisei* 算法地方大成 from 1838, a compass, protractor, and measuring square are depicted. The authors of this manual are Hasegawa Hiroshi 長谷川寛 (1782-1838) and Akita Hodo 秋田鳳堂 (dates unknown). Hiroshi was also an author of the 1833 mathematical text *Sanpo Tenzan Tebikigusa* 算法点竄手引艸 on the *tenzan jutsu* symbolic manipulation method along with Akita and Yamamoto Gazen 山本賀前 (1809 - ?). Hiroshi was also a co-author in an edition of the *Jinkōki* with Yamamoto called *Taizen Jinkoki* 大全塵劫記, as well as the popular *Sanpo Shinsho* 算法新書 with Chiba Tanehide 千葉胤秀 (1775 - 1849).

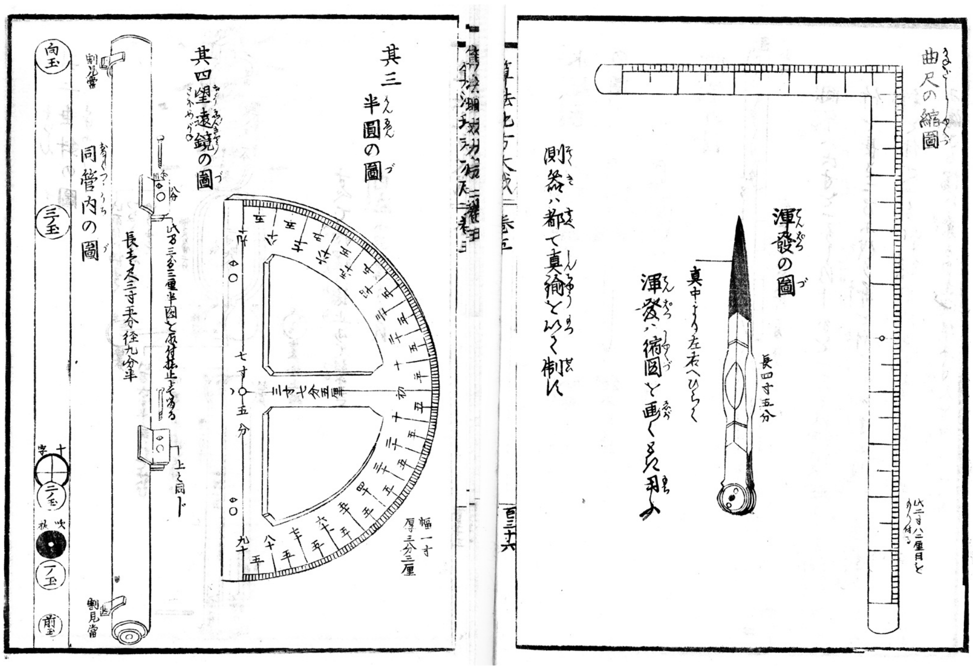


FIGURE 2.7: Protractor, compass, and square from 1838 CE *Sanpo Jikata Taisei*

Though he dabbled in land surveying, the main activity of Hasegawa was as a mathematician. The *Sanpo Jikata Taisei* deals with measurement, and instructs on how to find and measure angles. Seki Takakazu, considered the greatest mathematician of the Edo period, also is said to have written the text *Kikuyōmei Sanpo* 規矩要明算法 'Mathematical Manual Explaining the Compass and Square' [32, p. 124].

From a study of the works of Hasegawa and Seki, it can be seen that mathematicians were knowledgeable of measurement tools such as the protractor, compass, and square, and were most likely using them in the construction of diagrams. On *sangaku* themselves, there is also evidence that compasses were used for circles, as in some cases small dots appear in the middle of circles, suggesting the point of a compass.

While these were the tools for constructing and drawing mathematics, in the Edo period there were three main calculation tools available to mathematicians for calculating their problems. The first was the Japanese bead abacus known as the *soroban* 算盤. It is believed to be based off the Chinese *suanpan* 算盤 abacus which found its way to Japan from mainland China in the late 1500s. It consists of rows of beads in the base 10 system. The current *soroban* presents one bead with a value of five in its upper section, and four worth one in the lower (see Figure 2.8). The *soroban* was the most popular calculation tool of the Edo period, and even today it is still taught to primary school students in areas such as Nagano prefecture.

As well as this, the Japanese had a rod abacus (with rods called *sangi* 算木) which could be used for numerical calculation and to solve for one unknown. This tool was more common amongst the *samurai* class, who are said to have "despised the plebeian soroban" [87, p. 47]. This dislike for the device was likely driven by a wish to disassociate and distinguish themselves from the lower classes where *soroban* use was flourishing.

And, from the time of Seki Takakazu onwards, a form of symbolic manipulation known as *tenzan jutsu* 点竄術 which could solve for multiple unknowns was also available. This particular technique is discussed in detail in chapter 3.

FIGURE 2.8: *Soroban* and *sangi* counting rod abacus. (Images by author.)

It has been shown how the authors of *sangaku* came from all areas in Japanese society, as women, children, merchants, farmers, and *samurai* all produced these tablets. The majority would have learned mathematics from an *iemoto* style school, where they would obtain certification. Though, some more educated individuals could have been self taught. Before putting their work on a *sangaku*, mathematicians would use compass and straight edge tools to produce their diagrams, and lower quality paper for rough working. Depending on their class and level of education, different tools for doing the calculations themselves were also employed by mathematicians. The type of mathematics being done on *sangaku* is further explored in this chapter when a selection of unsolved *sangaku* problems are analysed and solved.

2.6 Numbers and Units of Measurement

Sangaku employ the traditional Japanese base ten counting system. The numbers from one to twenty in the Japanese style are listed in Table 2.3. In this system the largest value is placed before the smaller. For instance, 15 is written *jyu go* 十五, literally ‘ten five’. A value such as 20 is expressed *ni jyu* 二十 - ‘two ten’. For values in the hundreds and thousands, *hyaku* 百 and *sen* 千 are used in a similar manner. For example, 221 is *ni hyaku ni jyu ichi* 二百二十一 and 2221 is *ni sen ni hyaku ni jyu ichi* 二千二百二十一.

1	2	3	4	5	6	7	8	9	10
一	二	三	四	五	六	七	八	九	十
11	12	13	14	15	16	17	18	19	20
十一	十二	十三	十四	十五	十六	十七	十八	十九	二十 or 廿

Table 2.3: Japanese Number System

Between 1603 and 1868, the numbers 20 and 30 were also sometimes represented by specific characters. For example, while less common *ni jyu* 二十 appears as 廿 in some mathematical texts.

The Japanese also used units of measurement for lengths, widths, and volumes. Common units for lengths were *shaku* 尺 (1 *shaku* = 10 *sun*), *sun* 寸 (1 *sun* = 10 *bu*), *bu* 分 (1 *bu* = 10 *rin*), and *rin* 厘 (see Table 2.4). 1 *shaku* was roughly equivalent to 30.3 cm.

cho	*ken*	*shaku*	*sun*	*bu*	*rin*	*mō*	metric
1	60	36	3,600	36,000	360,000	3,600,000	109.1 m
	1	6	60	600	6,000	60,000	1.818 m
		1	10	100	1,000	10,000	30.30 cm
			1	10	100	1,000	3.030 cm
				1	10	100	0.303 cm
					1	10	0.0303 cm
						1	0.00303 cm

Table 2.4: Units for length

Decimal points were rarely given. The value of zero is expressed by the character *rei* 零, or sometimes with just a written ‘o’. When a zero appears in an answer, often the unit length is excluded or skipped. The character *ko* was also used on *sangaku* to represent units of rank 10 (see Table 2.5).

ko	*bu*	*rin*	*mō*	*shi*
1	0.1	0.01	0.001	0.0001

TABLE 2.5: Decimal Fractions

The units used for areas and volumes, along with how they can be expressed within the metric system, are listed in Tables 2.6 and 2.7.

se	*tsubo*	*go*	*shaku*	metric
1	30	300	3,000	99.17 m^2
	1	10	100	3.306 m^2
		1	10	0.331 m^2
			1	0.0331 m^2

TABLE 2.6: Units for Area

koku	*to*	*sho*	*go*	*shaku*	metric
1	10	100	1,000	10,000	180,400 cm^2
	1	10	100	1,000	18,040 cm^3
		1	10	100	1804 cm^3
			1	10	180.4 cm^3
				1	18.04 cm^3

TABLE 2.7: Units for Volume

2.7 Language Style

Sangaku were academic artifacts which employed a unique style of language. The language style of Edo period mathematical texts in general can be placed into three categories. The first of these is common Japanese using just known *kanji, hiragana*, and *katakana*. The Japanese language is made up of two alphabets, *hiragana* and *katakana* (see Tables 2.8 and 2.9). These alphabets represent Japanese sounds, which are either vowels or consonants followed by vowels.

Kanji are more complex characters derived from Chinese which are used in conjunction with *hiragana* and *katakana* to write Japanese. For instance, many verbs are created in Japanese using *kanji* characters as the stem and *hiragana* as the suffix. To change the tense of a verb, the *hiragana* is changed while the *kanji* stays the same. For example, *miru* 見る 'to see' combines the root *mi* 見 with the hiragana *ru* る. By changing *ru* る to *mashita* ました, the verb changes to the past tense *mimashita* 見ました 'saw'. *Katakana* is generally used to express new or foreign words, such as *kamera* カ

メラ for camera. An example of mathematics that uses this system of *kanji, hiragana*, and *katakana* is Yoshida Mitsuyoshi's *Jinkōki*. Yoshida's text was specifically aimed at instructing people with only basic reading abilities and learning.

あ a	か ka	さ sa	た ta	な na	は ha	ま ma	や ya	ら ra	わ wa
い i	き ki	し shi	ち chi	に ni	ひ hi	み mi		り ri	
う u	く ku	す su	つ tsu	ぬ nu	ふ fu	む mu	ゆ yu	る ru	
え e	け ke	せ se	て te	ね ne	へ he	め me		れ re	
お o	こ ko	そ so	と to	の no	ほ ho	も mo	よ yo	ろ ro	を wo
				ん n					

Table 2.8: Hiragana Alphabet

ア a	カ ka	サ sa	タ ta	ナ na	ハ ha	マ ma	ヤ ya	ラ ra	ワ wa
イ i	ケ ki	シ shi	チ chi	ニ ni	ヒ hi	ミ mi		リ ri	
ウ u	ク ku	ス su	ツ tsu	ヌ nu	フ fu	ム mu	ユ yu	ル ru	
エ e	ケ ke	セ se	テ te	ネ ne	ヘ he	メ me		レ re	
オ o	コ ko	ソ so	ト to	ノ no	ホ ho	モ mo	ヨ yo	ロ ro	ヲ wo
				ン n					

Table 2.9: Katakana Alphabet

The second category of language style is common Japanese which replaces *hiragana* with *katakana*. This is the style of Takebe Katahiro's 建部賢弘 (1664 – August 24, 1739) *Tetsujutsu Sankei* 秘術算経 'Mathematical Treatise on the Technique of Linkage', written for the *shōgun* Yoshimune Tokugawa 徳川吉宗像. *Katakana* often appears in texts designed for the *samurai* and upper classes, while *hiragana* is aimed at a more general audience including merchants and farmers. The third category is the academic language of *kanbun*, using or excluding *okurigana*. Since antiquity, a large number of key historical literary works have been written in the *kanbun*, including the oldest chronicles in Japan – the *kojiki* 古事記 (711) and *nihon shoki* 日本書紀 (720). Other important texts in *kanbun* include the Buddhist text *ōjō yōshū* 往生要集 (985) and poetry by authors such as Sugawa no Michizane 菅原道真 (845 - 903), Rai Sanyō 頼山陽 (1780 - 1832), and Natsume Sōseki 夏目漱石 (1867 - 1916) [101, p. 24]. Because of this, prior to the 1900s, *kanbun* was an vital part of training and education in Japan [101, p. 24].

The language is neither classical Chinese nor Chinese written by Japanese [44, p. xiii]. While originating from classical Chinese, over time the Chinese pronunciation of the characters began to be ignored, with Japanese words of the same or similar meaning adopted instead [44, p. 1]. This mix of Chinese and Japanese became *kanbun*. *Kanbun* ranges from pure Chinese characters to a mix of Chinese and Japanese. Added Japanese characters usually indicate particles and suffixes which are not expressed in Chinese

[44, p.2]. These are *okurigana*. *Okurigana* are markings beside *kanbun* characters that aid the reading of the characters in Japanese. The *okurigana* uses characters from the *katakana* alphabet. *Kanbun* was “the language of scholarship” [52, p. 26] of Japan, being similar to the use of Latin in Europe. It was “not easy…to read by those without a classical education” [12, p. 127], meaning the majority who wrote and read texts in this language were educated officials and *samurai*. This is because *samurai* were an educated class, with many of holding hereditary posts in administration and book keeping. The use of *katakana* in mathematical texts is often connected with members of the *samurai* class, with texts written with *hiragana* intended for merchants, farmers, and artistans. An example of a *kanbun* text is Seki Takakazu’s 1674 mathematical treatise *Hatsubi Sanpo* 発微算法, written in *kanbun* with only word order markings. However in his student Takebe’s commentary on this work, Takebe uses *kanbun* with *okurigana*.

The use of *kanbun* with and without *okurigana* on *sangaku* suggests that the practitioners creating these tablets were educated individuals who wished to have their work associated with the upper classes. Those creating tablets early in the tradition were most likely *samurai*, while those later may have been a mix of wealthy merchants and farmers as well as *samurai*. Their choice in language may have also been driven by the fact that *kanbun* was used by leading practitioners in the field, and thus considered the standard language for mathematical work.

2.8 Sangaku Problems

Having investigated the format, style, location, tools, and authors of *sangaku*, in this section I now examine some of their problems. I have considered nine tablets from throughout Japan, whose problems are translated into modern English and approached with contemporary mathematics. The problems contained on each *sangaku* are examined individually. Each problem is accompanied by a literal translation and modern analysis. Where possible textual and numerical breakdowns, as well as expressions in modern mathematical notation are provided. After illuminating the kind of mathematics that they contain, I further investigate traditional ways from the Edo period of solving *sangaku* problems in chapter 3 and question whether or not they should be considered just tools of communication, transmission, and promotion in chapter 4. Before examining these problems, a survey of the methods for their analysis is outlined below.

2.8.1 Presentation Notes

Each *sangaku* of this chapter is presented and examined through a series of sections displayed under the headings 1-6 below. Where multiple problems are presented on a tablet, headings 2-6 are repeated when necessary for each. Heading 1 is omitted when non-mathematical text is not available or has not managed to be translated. In cases where no technique section has been given by the *sangaku* author the technical analysis is omitted, as in this instance there is no procedure or formula to present.

1. **Accompanying Text**
   Text on the tablet which does not directly form part of the mathematical content is included separately before the mathematics is examined. Usually this text contains the name of the author, the school they were part of, the year of dedication, and the phrase 'dedication'.

2. **Translation**
   The mathematical text of each tablet is translated problem to problem. A transcription of the original text as it appears on the *sangaku* is provided first on the left-hand side, separated into the three problem, answer, and technique sections. An English translation is then given on the right-hand side of the sections. I attempt to keep the translation as true to the original Japanese as possible while formatting it to appeal to the English reader.

3. **Translation Notes**
   Where text appears that is divergent from the standard forms discussed in this chapter, it is illuminated and explained before the mathematics is analysed.

4. **Technical Analysis**
   To elucidate the mathematical problems more clearly, the problem text is rendered in the following two manners:

   Rendering 1 - Procedure

   The first rendering gives the problem in plain English. The technique section is broken down literally, with each step presented on a separate line. The numerical value produced by the calculations is also listed on an individual line. This is done to preserve the procedural aspect of the text, and to illustrate the original presentation given by the author.

   On *sangaku*, often a series of calculations may be given a label. Where the author gives such a label, it is expressed in the procedure by means of a right arrow $\rightarrow$ to indicate the operations are now classified by a specific variable. For example, the text 'name *heaven*' appears on some of the examined tablets. This would be expressed as [series of calculations] $\rightarrow$ *heaven*

All following occurrences of the label *heaven* would refer back to this series of calculations in the technique section. Further discussion of these labels is given in section 2.8.5.1.

RENDERING 2 - FORMULA

The second rending of the text translates figures of the diagram into variables such as a, b, c, d, etc, and presents the technique section in the form of a single formula. This presents the mathematical problem in a more familiar way for the modern reader which is easier to compare with results from the modern analysis section. Where variables such as '*heaven*' appear, their value in terms of the calculations they represent is shown first before the display of the formula.

5. **Modern Analysis**
   Given the absence of working for the solution provided on the tablet, each problem is approached and solved using modern mathematical methods. These are related back to *Rendering 2* where possible to show how the formula may have been obtained.

6. **Comments**
   Additional elements of the tablet which provide interesting insights into the tradition are discussed in this section after the problem has been examined.

2.8.2 Translation Method Notes

To translate *sangaku*, I have connected *kanbun* characters found on *sangaku* back to related modern Japanese mathematical terminology. For example, the term *ka* 加 which appears on tablets means 'addition' in modern day Japanese. It can be assumed that the *sangaku* term *ka* 加 is related, and by treating this character as 'addition' on tablets it can be seen that this is a correct interpretation, for it produces the values on tablets. Through similar investigative work mathematical operators on *sangaku* have been determined. As well as this, I have made use of Morimoto and Ogawa's list of terminology in the appendix to *Mathematical Treatise on Technique of Linkage: An Annotated English Translation of Takebe Katahiro's Tetsujutsu Sankei*. Other terminology has been determined through the use of general *kanbun* dictionaries, which give Japanese translations of *kanbun* terms which can be then translated into English, and modern Japanese translations of *sangaku* texts which are given in many articles. From the amalgamation of these sources the translations in this chapter and thesis have been derived.

The language of *sangaku* is quite conceptual, with often just the core nouns and verbs present. For example, in the Enman-ji tablet to be examined, a section of text reads ‘triangle inside hexagon’ 三角内六角. Broken down, the words used are *sankaku* 三角 for triangle, *nai/uchi* 内 meaning ‘within’, and *rokkaku* 六角 meaning hexagon. This literally reads ‘triangle inside hexagon’, describing a hexagon inside a triangle. In modern Japanese one could express ‘inside the triangle there is a hexagon’ using the sentence 三角の内に六角があります. This includes the core words from the *kanbun* sentence - triangle, inside, hexagon - but gives qualities of existence (‘there is’) and defines which figure is in what, for from just ‘triangle inside hexagon’ without a diagram it is unclear which figure is inside which. Because of its conceptual nature, the language of *sangaku* can come across as vague. To make the text clearer I do not translate it strictly literally, and attempt to give a translation true to the text which is at the same time understandable and appealing for the English reader.

2.8.3 Key Terminology

Some of the common terms which appear in the text of *sangaku* are listed below for the benefit of the reader.

Mathematical Operators

Jyō	乘	Multiplication
Jishi, jino	自之	Square (lit. self multiplication)
Jijyō	自乘	Square
Beki	冪	Square, power of
Kaiheihō	開平方	Take square root
Gen	減	Subtraction
Jyo	除	Division
Ka	加	Addition
Bai	倍	Double
Hanno, *hankore*	半之	Halve

Figures and Others

Sankaku	三角	Triangle
Rokkaku	六角	Hexagon
En	圓/円	Circle
Hanen	半圓/円	Semi-circle
Kata	方	Square

Tei	梯	Trapezium, trapezoid
Hishi	菱	Diamond, rhombus, parallelogram
Ya	矢	Sagitta (lit. arrow)
Uchi, nai	内	Inside, within
Man	面/面	Side (length)
Kei	徑	Diameter
Hasu	斜	Line, diagonal
Mon, toi	問	Problem, question
Jutsu-ni-iwaku	術日	Technique
Kotae-ni-iwaku	答日	Answer
Tadashi/tadaiu	只云/只言	Say
Yuki	有奇	Indicates an approximation

2.8.4 Indication of Approximation

In some instances, after a numerical value has been provided in the answer section the characters *yuki* 有奇 appear. These indicate that the numerical answer given is an approximation, and there are further values in the answer which the author has not chosen to include. Where these characters appear in a *sangaku* text, following the number in the translation are a series of dots ... to indicate further figures. For example, the Yoshifuji Mishima tablet (section 2.8.8) answer section includes the text *go jyu san ken yuki* 五十三間有奇, which is translated as "53 ... *ken*".

2.8.5 Letters and Labelling

Kanji	Japanese	Reading	Kanji	Japanese	Reading
甲	*kō*	1st	大	*dai*	large
乙	*otsu*	2nd	中	*chū*	medium
丙	*hei*	3rd	小	*shō*	small
丁	*tei*	4th			
戊	*bo*	5th	上	*ue*	upper
己	*ki*	6th	下	*shita*	lower
庚	*kō*	7th	外	*soto*	outer
辛	*shin*	8th	等	*tou*	equal
壬	*jin*	9th	平	*hira*	flat, ordinary
癸	*ki*	10th	全	*zen*	all

Table 2.10: Common Labels Used on Sangaku

There are two main labelling systems used for figures on *sangaku*. The first uses characters from the Chinese calender (see Figure 2.10) such as *kō* 甲, *otsu* 乙, *hei* 丙, and *tei* 丁. These characters are treated as ordinal numbers and can be thought of as akin to *a, b, c, d, etc* or first, second, third etc [15, p. 260-1]. This labelling system is still used instead of letter grades on Chinese exam papers [15, p. 260-1]. On *sangaku*, these characters are generally are applied in order to figures starting from the largest. The largest is usually labelled *kō* 甲, the next largest *otsu* 乙, and so on.

The second method is to label figures according to their size with *dai* 大 the largest, *chū* 中 the medium value, and *shō* 小 the smallest. When characters from the Chinese calender are used to label figures, since they do not have a direct English translation, the original character and its pronunciation in English are inserted into the text. For instance, if there is a circle labelled with 甲 on the diagram, this is translated in the transcription as *kō* 甲. If a figure is labelled using the *dai* 大 large, *chū* 中 medium, *shō* 小 small system, the English translation is first used with the original character in brackets 'large (大)', then after all instances of the character are referred to by the English translation. This is done to keep as close as possible to the original reading intended by the author.

Circles containing other circles are also sometimes labelled *soto* 外, which means 'outside'. There are also instances where circles are labelled *tou* 等 meaning 'equal' to indicate circles labelled with this character are the same size. These are treated in the same manner as *dai* 大, *chū* 中, and *shō* 小, with the first instance having the English translation with the character in brackets - 'outer circle (外)', 'equal circles (等)' - and all other instances having just the English. The characters *zen* 全 and *hira* 平 are also used on occasion to label circles. The character *zen* 全 translates to "all", and *hira* 平 to either "flat" or "ordinary". Because using 'all' and 'flat' as labels may be confusing in English translations, these are treated like the characters from the Chinese calender with the character and its reading in English included in the text when they appear. Lines identified based on their position with the characters *ue* 上 'upper' and *shita* 下 'lower' use just the English translations 'upper' and 'lower'.

2.8.5.1 Series of Operations - Pole, Heaven, Earth

As discussed in section 2.8.1, *sangaku* authors often label series of operations in the technique section with terms. The common labels are *kyoku* 極 'pole', *ten* 天 'heaven', and *chi* 地 'earth'. For example, a section of text from the Isaniwa *sangaku* examined in section 2.8.9 reads "Combine the diameter of circle *otsu* 乙 and the diameter of circle *hei* 丙. Name this the pole". After the calculation *otsu* 乙 + *hei* 丙 has been given the label *pole*, any instances where the term *pole* appears the calculation *otsu* 乙 + *hei* 丙 is inferred. Through labelling series of calculations by variables in this way, the author is able to make the technique section more concise and clear.

I have translated the character *kyoku* 極 as 'pole', but the *kanbun* dictionary *Kanbunpou Kiso - Hontou ni Wakaru Kanbun Nyumon* 漢文法基礎本当にわかる漢文入門 also translates (in English) the character as "extreme, highest, topmost, farthest, utmost, extremely" [61, p. 319]. It is found in terms such as *gokkan* 極寒 'extreme cold', *kiwamaru* 極まる 'to reach the end', 'to reach an extreme', and *kyokuiki* 極域 'polar'. This particular phrasing may have been used because it is intended to represent the value or result at the end of a series of operations, so it could be thought of as the end or climax of the operations. The terms *ten* 天 'heaven' and *chi* 地 'earth' also function in a similar manner to *kyoku* 極 'pole'. Where there are two sets of calculations the author wishes to label, the terms *ten* 天 'heaven' and *chi* 地 'earth' usually appear. This may be because together they also express the same sense and imagery of extremes.

2.8.5.2 Non-numerical Values and Answers

In some instances the characters *jyakkan* 若干 appear instead of numerical values on the diagram. For example, the problem section may contain text such as "large circle diameter *jyakkan* 若干". Here *jyakkan* 若干 indicates that while the actual numerical value of a figure is not given, one is to find the solution *in terms* of that particular figure. Due to this I translate *jyakkan* 若干 as "is known". This practice first appeared in the work of the mathematician Seki Takakazu, in his 1674 *Hatsubi Sanpo* 発微算法.

In some cases 依左術 or 合左術 'technique on the left' is also written instead of a numerical value in the answer section. This directs the observer towards the formula in the technique section for the solution. This indicates there are two different ways that problems present on *sangaku*. Firstly, some seek a specific figure in terms of another. Secondy, there are some which seek a numerical answer, though it is obtained by putting a figure in terms of another and then applying numerical values given in the problem section to obtain a numerical result.

While both types essentially have the same types of formula sections (for the formula section is always where one figure is put in terms of others), the exclusion of numerical values in the first type gives an additional level of abstraction to the problem. For, the answer does not have a numerical value which the reader can check against or work backwards from; all they have to rely on is a deep understanding of the geometry. The language used on *sangaku* thus emphasises understanding the relationships between figures given in the diagram. It is through this understanding that the way to solve the problem could be found.

2.8.5.3 Right Angled Triangles

Right angle triangles have their own specific labelling convention. The characters *kou* 勾, *ko* 股, *gen* 弦 consistently appear connected with specific side lengths of right angle triangles. In the Japanese tradition, the shortest side of the triangle is labelled *kou* 勾, the second longest side *ko* 股, and the hypotenuse *gen* 弦. Edo period textbooks such as the *Sanpo Shinsho* and *Taizen Jinkoki* place computations concerning right angle triangles in a chapter labelled *koukogen* 勾股弦. In modern Japanese, the word *koukogen* 勾股弦 can be translated as the Pythagorean theorem.

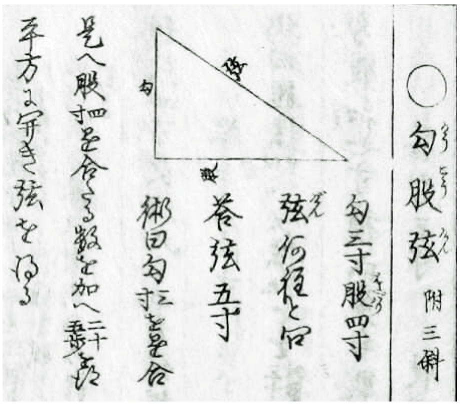


FIGURE 2.9: Right angle triangle problem dealing with Pythagorean theorem from *Sanpo Shinsho*

The Japanese *koukogen* 勾股弦 appears to have been adapted from the Chinese word *gougu* or *kouku* 勾股 translated in the modern day as the Pythagorean theorem. Dauben explains that "in ancient Chinese there was no word (and apparently no mathematical concept) of triangle per se…Instead, Chinese geometry speaks of the *gou* (or base/shadow), the *gu* (or height/gnomon), and *xian* (the hypotenuse that joins the extreme edge of the *gou* and *gu*)" [13, p. 215]. This learning was transmitted to Japan via Chinese mathematical texts, and the use of the terms throughout the Edo period indicates the concept did not change.

The terms 勾股弦 encompass the concept of short side, long side, and hypotenuse of a right angle triangle as well as the idea of a right angle triangle as a whole. They also signal the Pythagorean theorem. In chapter 3, mathematical rules of the Japanese symbolic manipulation technique will be given. One rule lacking from the collection is the Pythagorean theorem. This may be because the rule was already tied up in the concept of a right angle triangle. When *kou* 勾 appears, I translate this as 'short side'. Similarly, I translate *ko* 股 as 'long side' and *gen* 弦 as 'hypotenuse'. The reader should bear in mind that when reading the problems that appear later in this work (particularly in chapter 3) these terms and right angle triangles are associated with the Pythagorean theorem in the Japanese tradition. Due to this the use of the theorem may be indicated without explicit reference by a right angle triangle appearing in a problem with these characters.

2.8.6 Enman-ji Sangaku

FIGURE 2.10: The *sangaku* at Enman-ji temple, Nara prefecture. (Image by author).

Enman-ji 円満寺 is a small temple located in the semi-rural town of Yamacho in Nara prefecture. The characters forming its name *enman* 円満 mean harmony, peace, perfection, and integrity, and the temple enshrines the Japanese deity of protection and culture *Hachiman*. Enman-ji contains one votive mathematical tablet dedicated by Tōmura Genirō 源治郎 which presents three problems dealing with hexagons, triangles, circles, and squares. The original tablet was dedicated in 1844. A replica was placed at the temple by the Nara City Board of Education in 1994. On a plaque underneath the Enman-ji tablet, also written and placed by the Nara city Board of Education, the following information about the author is given:

> The dedicator Tōmura Genirō, of our hometown, was a person who studied wasan, and it is thought that he dedicated the results of his private wasan learning as a sangaku.

From this it appears the author was a local of the Yamacho area and was largely self-taught in mathematics. This particular tablet, as will be elaborated, varies from others in the tradition in its use of coefficients in the technique sections for problems one and two, and through the inclusion of unique symbols to mark different parts of the second problem.

2.8.6.1 Sources and Transcription

The problems of this tablet have not been previously transcribed and solved to date within the literature. I have produced my transcriptions based off photographs personally taken of the tablet in Japan in 2012. See D.1 in Appendix D for a larger image.

2.8.6.2 Accompanying Text

Before Problems

奉納御寶前	Dedication - Before the honorable treasure.

After Problems

當村	Tōmura
願主　源治郎	Temple petitioner - Genjirō
天保十五年甲長年仲夏	Tempō era 15th year (1844). Year of the Wood Dragon. May/Midsummer

2.8.6.3 Enman-ji: First Problem

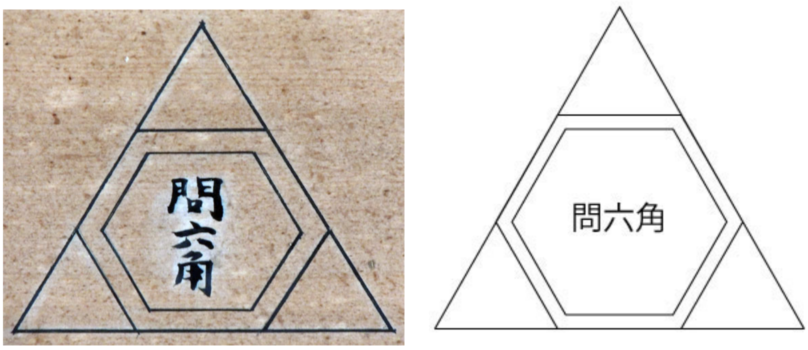


Figure 2.11: Left: First Enman-ji problem. Right: Transcription. (Image by author).

Translation

TRANSCRIPTION	TRANSLATION
今有如圖三角内六角入 三角面六尺問六角面	As in the diagram, there is triangle with a hexagon inscribed inside. The side length of the triangle is 6 *shaku*. Problem - side length of the hexagon.
答日　六角一方二尺	Answer: The side length of the hexagon is 2 *shaku*
術日面六尺自之 四三三乘三除術減止 余二九八以除是止五 余開平方得合問	Technique: Square the side length of 6 *shaku*. Multiply by 433, then divide by 3. Subtract this and stop. Divide the remainder by means of 2598. Stop. Take the square root of the remainder. Obtain [the hexagon side length] as required.

Translation Notes

In this problem, the author shows that when there is a large equilateral triangle with sides of 6 *shaku*, there will be a hexagon inside it which is 2 *shaku* in side length. However while the text only refers to one hexagon, two are present in the diagram in Figure 2.11. The innermost hexagon on the diagram has the text *mon rokkaku* 問六角 “problem hexagon” written inside it, but reasoning from the numerical values supplied for the triangle and hexagon shows the outer hexagon is the figure sought. The reasoning required to determine which hexagon is the problem in itself produces the answer, for it is one third the side of the triangle (with one third of 6 being 2). However the author employs a more complicated formula to obtain this numerical answer.

A feature of this text which diverges from others in the tradition is the appearance of values without units of measurement in the technique section. These are the numbers 433 (*shi san san* 四三三) and 2598 (*ni go kyu hachi* 二五九八). The value 433 derives from the Edo period ‘regular triangle rate’ (*sankaku no hō* 三角の法). This rate first appears in the Japanese tradition with relation to equilateral triangle area calculations in the following passage of Yoshida Mitsuyoshi’s 吉田光由 (1598-1673) *Jinkōki* 塵劫記 (1627), the earliest and most popular mathematical textbook of the Edo period

> Find the area of an equilateral triangle rice field of which each side is 15 *ken* long.
>
> 3 *se* 7 *bu* 425.

> <Process> Make the square of 15 *ken*. You get 225 *bu*. Multiply it by the regular triangle rate 433, and you get 97 *bu* 425. Then divide it by the field area rate, 3.[4] And you get the answer [55, p. 94-5].

The regular triangle rate can be understood as an approximation the numerical value 0.4330127019, which can be expressed as the modern day fraction $\frac{\sqrt{3}}{4}$ or $\frac{1}{2}\sin 60°$. The value 2598 also appears in the *Jinkōki* as part of the 'hexagon rate' for calculating the area of hexagons [55, p. 94]. With regard to hexagons, the *Jinkōki* presents a problem where the area of a hexagon is found by squaring the side length, multiplying by the regular triangle rate 433, and then multiplying by 6. This essentially treats the hexagon as made up of 6 small equilateral triangles, which is why the triangle rate is multiplied by 6 ($\frac{\sqrt{3}}{4} \times 6 = 2.598$). Values like 433 and 2598 without place value indicators or accompanying units of measurement reference can be understood as relating to specific rates such as these, and act as coefficients.

Technical Analysis

The author shows that the (outer) hexagon has a side length of 2 *shaku* through a series of calculations pertaining to areas. They obtain the area of the hexagon by determining the area of the large triangle and then subtracting a third of this area to leave just the area of the hexagon. Then from this the side length squared of the hexagon is found, and the square root of this value taken to produce the answer.

Rendering 1 - Procedure

Problem Say the triangle's sides are 6 *shaku*. What is the length of the hexagon's sides?

Answer Hexagon side = 2 *shaku*

Solution

$$\text{Triangle side}^2 \times 433$$

$$(\text{Triangle side}^2 \times 433) \div 3$$

$$(\text{Triangle side}^2 \times 433) - ((\text{triangle side}^2 \times 433) \div 3) = 10392$$

$$10392 \div 2598$$

$$\sqrt{10392 \div 2598} = \text{hexagon side}$$

Hexagon side = 2 *shaku*

Rendering 2 - Formula

[4]This field area rate division is due to the fact that 1 *se* = 30 *bu*

Let the triangle side length $= s$ and the hexagon side length $= x$.

Problem Say $s = 6$. What is x?

Answer $x = 2$

Solution $$x = \sqrt{\frac{\left(s^2 \times \frac{\sqrt{3}}{4}\right) - \left(\left(s^2 \times \frac{\sqrt{3}}{4}\right) \div 3\right)}{6 \times \frac{\sqrt{3}}{4}}}$$

Modern Analysis

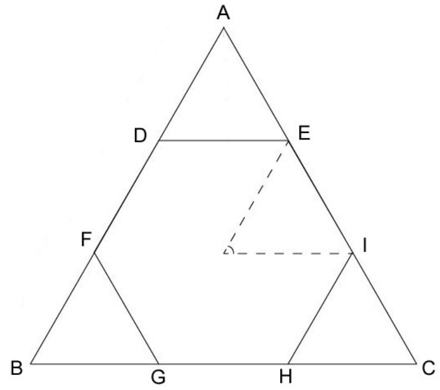


FIGURE 2.12: Enman-ji: First problem analysis

As in Figure 2.12, let $\triangle ABC$ be the large triangle. Assume triangle $\triangle ABC$ consists of three smaller equilateral triangles $\triangle ADE$, $\triangle BFG$, $\triangle CHI$, and a hexagon.

Using the area formula *side length*$^2 \times \frac{\sqrt{3}}{4}$ to obtain the area of $\triangle ABC$

$$\left(6^2 \times \frac{\sqrt{3}}{4}\right) = 15.588$$

Dividing the area of $\triangle ABC$ by three - with each third equal to the combined area of $\triangle ADE + \triangle BFG + \triangle CHI$. Then subtracting this area of $\triangle ADE + \triangle BFG + \triangle CHI$

from the area of the triangle $\triangle ABC$. This gives the area of the hexagon

$$\left(6^2 \times \frac{\sqrt{3}}{4}\right) \div 3 = 5.196$$
$$15.588 - 5.196 = 10.392$$

Then multiply the triangle area rate by six, and divide into the hexagon area to find the side length of the hexagon

$$10.392 \div \left(6 \times \frac{\sqrt{3}}{4}\right) = 4$$
$$\sqrt{4} = 2 \tag{2.1}$$

2.8.6.4 Enman-ji: Second Problem

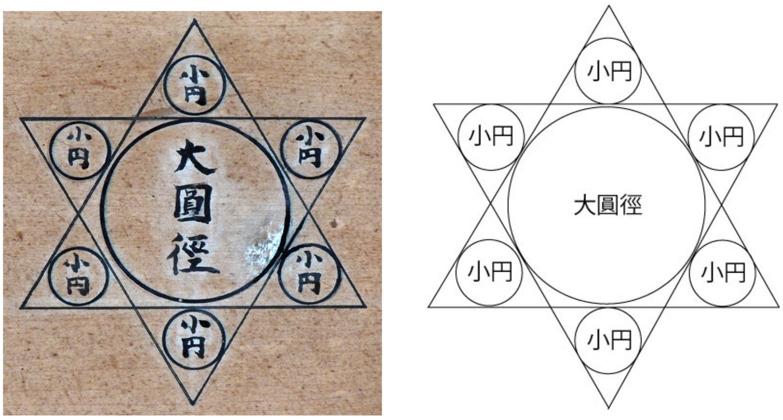


FIGURE 2.13: Left: Second Enman-ji problem. Right: Transcription. (Image by author).

Translation

Transcription	Translation
〽[5]今有如圖六角面三尺 内大圓徑入 〽[6]又有外角小円徑入	As in the diagram, there is a hexagon with sides of 3 *shaku* inscribed inside the large circle (大). And, as well as this, inthe outside corners are inscribed small circles (小).
答云 大円徑五尺一寸九分六厘 小円徑一尺七寸三分二厘	Answer The diameter of the large circle is 5 *shaku* 1 *sun* 9 *bu* 6 *rin* The diameter of the small circle is 1 *shaku* 7 *sun* 3 *bu* 2 *rin*
〽[7]術日面三尺自乘二五九八乘 三除實面半之以實除円得合問 〽[8]又日面三尺自之四三三乘 以三除其上以七五除是得合問	Technique: Square the side length of 3 *shaku*. Multiply by 2598. Divide by 3. Divide by the length of half a side. Obtain the circle as required. As well as this, square the side length 3 *shaku*. Multiply by 433. Divide by means of 3. Divide the previous by means of 75. Obtain [small circle] as required.

Translation Notes

Interesting symbols, not seen elsewhere, appear on this tablet to distinguish specific sections of the text. The text related to first problem is marked by the symbol shown left in Figure 2.14. The section of text pertaining to the second problem is marked by the symbol on the right.

FIGURE 2.14: Characters marking different sections

[4]The character 〽 is not found in and of itself in the Japanese language, and is a combination of a circle with the part alternation mark 〽. Only online sources comment on the mark, which believe it was used to indicate the start of a new verse in song or part in a Noh play. The mark is found in modern Japanese as the unicode character U+303D. Given the context of the text, it appears the circle is used with the mark to indicate the beginning of the text related to the circle in the diagram.

[5]This character combines the part alternation mark with a triangle. Given the context of the text, it appears to mark the start of the problem relating to the circles within the small triangles in the diagram.

[6]The part alternation mark and circle are used here again to denote the following text pertains to the circle problem.

[7]The part alternation mark and triangle are used to indicate that the text relates to the circles in the small triangles.

The use of symbols to label different parts of a problem is rare in the *sangaku* tradition, and other *sangaku* texts and mathematical treatises of the time that I have seen do not appear to use them. They function as a visual aid to separate the two problems of the text. This may indicate that the author wanted the problem to be as clear and understandable as possible to the observer, showing a great desire for the tablet to be viewed, understood, and solved.

Technical Analysis

The author finds the value of the large circle diameter within the middle of the hexagon by first finding the area of the hexagon and dividing it by 3, producing $\frac{1}{3}$ of the area. As mentioned, in the Japanese tradition texts such as the *Jinkōki* consider hexagons as made up of six equilateral triangles. The author treats the hexagon in this manner, seeing one third of its area as equivalent to the area of two smaller triangles. With this in hand, they use the height of these two triangles - which are on top of one another vertically and equivalent in height to the circle diameter sought - and divide by half the base of one of the triangles (the same as half the side length since they are equilateral), producing the answer of 5 *shaku* 1 *sun* 9 *bu* 6 *rin*.

For the small circles, the author first finds half the altitude of the small triangles that inscribe them by multiplying the triangle rate 433 by the side length of the hexagon. Where h is the altitude, the author divides by 3 to produce $\frac{1}{2}h \div 3 = \frac{1}{6}h$. Since one third the altitude of the triangle ($\frac{1}{3}h$) is equivalent to half the diameter of the inscribed circle ($\frac{1}{2}d$), and therefore $\frac{1}{6}h$ is equivalent to $\frac{1}{4}d$, this gives $3 \times \frac{1}{4}d$ which is $\frac{3}{4}d$. The author then divides by 75 - equivalent to $\frac{3}{4}$ - leaving the diameter of the circle (since $\frac{3}{4}d \div \frac{3}{4} = d$).

Rendering 1 - Procedure

Problem Say the hexagon's sides are 3 *shaku*. What are the diameters of the large and small circles?

Answer Large circle = 5196

Small circle = 1732

Solution Hexagon side^2 × 2598

(Hexagon side^2 × 2598) ÷ 3

(Hexagon side^2 × 2598) ÷ 3) ÷ 1.5 = large circle

Large circle = 5196 *sun*

Hexagon side^2 × 433

(Hexagon side^2 × 433) ÷ 3

((Hexagon side^2 × 433) ÷ 3) ÷ 75 = small circle

Small circle = 1732 *sun*

Rendering 2 - Formula

Let the side of the hexagon = s, the diameter of the large circle = x, and the diameter of the small circle = y.

Problem Say $s = 2$. What are the values of x and y?

Answer $x = 5.196$

$y = 1.723$

Solution $$x = \frac{\frac{1}{3}\left(3^2 \times \frac{3\sqrt{3}}{2}\right)}{\frac{3}{2}} = 3\sqrt{3} \qquad y = \frac{\frac{1}{3}\left(3^2 \times \frac{\sqrt{3}}{4}\right)}{\frac{3}{4}} = \sqrt{3}$$

Large Circle: Modern Analysis

As in Figure 2.15, let A be the centre of the large circle, AB its radius, and s the side length of the hexagon, and r the radius of the small circles.

Using the author's method, the area of the hexagon is calculated using s and the hexagon rate. This treats the hexagon as made up of six triangles whose areas are found and combined to express the hexagon area

$$3^2 \times \left(\frac{\sqrt{3}}{4} \times 6\right) = 23.382$$

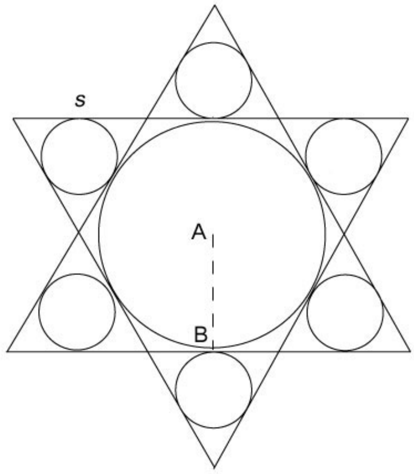


FIGURE 2.15: Enman-ji: Second problem analysis

Then this area is divided by three to leave two of the six triangles seen as making up the hexagon

$$23.382 \div 3 = 7.794$$

Lastly this is divided by half the base of the triangles to find the answer

$$7.794 \div \frac{3}{2} = 3\sqrt{3} = 5.196 \tag{2.2}$$

Small Circle: Modern Analysis

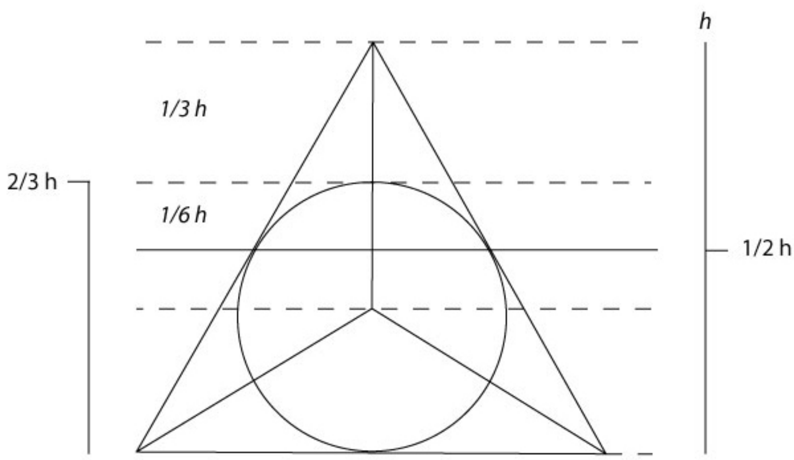


FIGURE 2.16: Enman-ji: Second problem analysis cont.

The altitude of the triangle is first found through multiplying the side length squared by the triangle rate

$$3 \times \left(3 \times \frac{\sqrt{3}}{4}\right)$$

Then the altitude is divided by three, and one third of the altitude is subtracted to leave the diameter of the circle

$$\left(3 \times \left(3 \times \frac{\sqrt{3}}{4}\right)\right) \div 3 = 3 \times \frac{\sqrt{3}}{4} = \frac{3}{4} \cdot \sqrt{3}$$

$$\frac{3}{4} \cdot \sqrt{3} \div \frac{3}{4} = \sqrt{3} \left(\frac{3}{4} \cdot d \div \frac{3}{4} = d\right) \tag{2.3}$$

2.8.6.5 Enman-ji: Third Problem

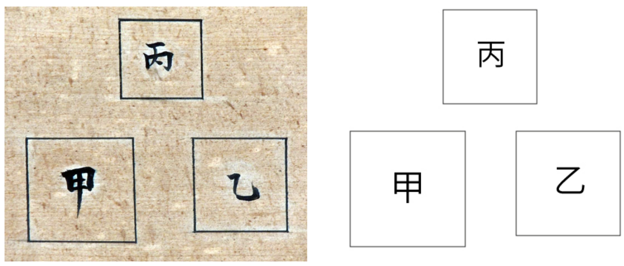


Figure 2.17: Left: Third Enman-ji problem. Right: Transcription. (Image by author).

Translation

Transcription	Translation
今有如圖方面三和甲乙方寸巾二和七百二十四坪乙丙方寸巾和五百八十坪差二寸定甲乙丙方寸問	As in the diagram, there are three squares. The sum of *kō* 甲 squared and *otsu* 乙 squared in *sun* is 724 *tsubo*. The sum of *otsu* 乙 squared and *hei* 丙 squared in *sun* is 580 *tsubo*. The set difference of the sides of *kō* 甲, *otsu* 乙 and *hei* 丙 is 2 *sun*.
答日	Answer
甲方寸二尺	The side in *sun* of the square *kō* 甲 is 2 *shaku*
乙方寸一尺八寸	The side in *sun* of the square *otsu* 乙 is 1 *shaku* 8 *sun*
丙方寸一尺六寸	The side in *sun* of *hei* 丙 is 1 *shaku* 6 *sun*

術日差二寸自乘是半内之七百二十四坪減止余半之開平方是倍是差加半之甲方寸丙差倍減得又乙差減得	Technique: From the difference of 2 *sun*, multiply by half of this. By 724 *tsubo* subtract. Halve and take the square root. Double and now add the difference and half for *kō* 甲 in *sun* squared. Double and subtract the difference to obtain *hei* 丙. Also subtract the difference to obtain *otsu* 乙 [as required].

Translation Notes

This problem asks the reader to find the individual values of the three squares in the diagram (Figure 2.17) when just the sum of two of the squares at a time and the difference between them is known.

The author gives the combined value squared for the squares *kō* 甲 and *otsu* 乙 as 724 *tsubo*. The measurement unit *tsubo* is for area and expresses area in terms of *ken* squared. For instance, one *tsubo* equates to roughly 3.95 square yards in modern terminology [99, p. 180]. In Japan, one may ask how many *tsubo* a property is instead of square feet or yards [60, p. 120]. The term encompasses the squaring of the squares *kō* 甲 and *otsu* 乙 within its meaning. The characters 差二寸〔二〕are used in the text to express a difference in value, where the character *sa* 差 indicates difference, giving a reading of "difference of 2 *sun*".

While 1 *tsubo* usually equals 6 *shaku*^{2}, in this instance the author states that the values 724 and 580 are in *sun*^{2}. This allows for the answer to be expressed in terms of *shaku* and *sun* rather than *ken*.

Technical Analysis

Rendering 1 - Procedure

Problem Say the sum of *kō* 甲 and *otsu* 乙 squared is 724 *tsubo*, and the sum of *otsu* 乙 and *hei* 丙 squared is 580 *tsubo*. When the difference between each square is 2 *sun*, what are their individual values?

Answer Square *kō* 甲 in *sun*^{2} = 2 *shaku*

Square *otsu* 乙 in *sun*^{2} = 1.8 *shaku*

Square *hei* 丙 in $sun^2 = 1.6$ *shaku*

Solution $2 \times \frac{2}{2}$

$724 - (2 \times \frac{2}{2})$

$\sqrt{\left(724 - \left(2 \times \frac{2}{2}\right)\right) \div 2}$

$\left(\left(\sqrt{\left(724 - \left(2 \times \frac{2}{2}\right)\right) \div 2}\right) \times 2\right) + 2$

$\left(\left(\left(\sqrt{\left(724 - \left(2 \times \frac{2}{2}\right)\right) \div 2}\right) \times 2\right) + 2\right) \div 2 = k\bar{o}$ 甲

$k\bar{o}$ 甲 $= 2$ *shaku*

$\left(\left(\left(\left(\sqrt{\left(724 - \left(2 \times \frac{2}{2}\right)\right) \div 2}\right) \times 2\right) + 2\right) \div 2\right) - 4 = hei$ 丙

hei 丙 $= 1.8$ *shaku*

$\left(\left(\left(\left(\sqrt{\left(724 - \left(2 \times \frac{2}{2}\right)\right) \div 2}\right) \times 2\right) + 2\right) \div 2\right) - 2 = otsu$ 乙

$otsu$ 乙 $= 1.6$ *shaku*

Rendering 2 - Formula

Let the square *kō* 甲 $= x$, *otsu* 乙 $= y$, *hei* 丙 $= z$, and the difference between them $= d$.

Problem Say $x^2 + y^2 = 724$, and $y^2 + z^2 = 580$. When $d = 2$, what are the individual values of x, y, and z?

Answer $x = 2$

$y = 1.8$

$z = 1.6$

Solution $x = \dfrac{2\left(\dfrac{724 - (d \times \frac{d}{2})}{2}\right) + 2}{2}$ $y = \left(\dfrac{2\left(\dfrac{724 - (d \times \frac{d}{2})}{2}\right) + 2}{2}\right) - d$

$$z = \left(\frac{2\left(\dfrac{724 - (d \times \frac{d}{2})}{2}\right) + 2}{2}\right) - 2d$$

Modern Analysis

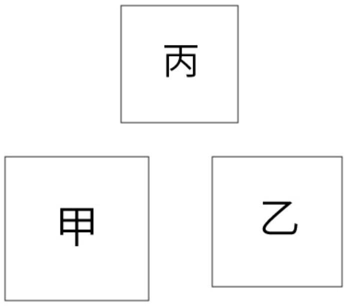


Figure 2.18: Enman-ji: Third problem analysis

Where *kō* 甲 is represented as x and *otsu* 乙 as y, the text reads that x^2 and y^2 added together give 724 *tsubo*. There is a difference of 2 *sun* between them, such that $724 = (x + y)^2 = y^2 + (y + 2)^2$. This can be used to find the perfect square of 19^2. This produces $y + \frac{1}{2} \cdot d = 19$ (where $d =$ the difference of 2 *sun*), allowing y and $(y + 2)$ to then be solved.

Where 甲 is represented as x, and 乙 as y,

$$
\begin{aligned}
&724 = x^2 + y^2 \\
&x = y + 2 \\
&x^2 + y^2 = y^2 + (y+2)^2 \\
&y^2 + (y+2)^2 = 2y^2 + 2(y \cdot 2) + 2^2 \\
&724 - 2 = (2y^2 + 2(y \cdot 2) + 2^2) - \frac{2^2}{2} \\
&(2y^2 + 2(y \cdot 2) + 2^2) - \frac{2^2}{2} = 2y^2 + 2(y \cdot 2) + \frac{1}{2} \cdot 2^2 \\
&\frac{724 - 2}{2} = y^2 + (y \cdot 2) + \frac{1}{4} \cdot 2^2 \\
&\sqrt{\frac{724 - 2}{2}} = \sqrt{(}y^2 + (y \cdot 2) + \frac{1}{4} \cdot 2^2) = y + \frac{1}{2} \cdot 2 = 19 \\
&(y + \frac{1}{2} \cdot 2) \times 2 = 2y + 2 = 19 \times 2 = 38 \\
&38 + 2 = (2y + 2) + 2 = 40 \\
&\frac{2y + 4}{2} = y + 2 = 20 \\
&x = 20
\end{aligned} \tag{2.4}
$$

Additionally for y and z

$$
\begin{aligned}
&(2y + 2) + 2 = (2y + 4) = 40 \\
&(2y + 4) - 4 = 2y \\
&2y = 40 - 4 = 36 \\
&\frac{2y}{2} = \frac{36}{2} \\
&y = 18 \\
&580 = 18^2 + z^2 \\
&580 - 18^2 = z^2 \\
&z = 16
\end{aligned} \tag{2.5}
$$

2.8.6.6 Comments

From examining the Enman-ji *sangaku*, it can be seen that the author had a reasonably high level of education given their knowledge of the *kanbun* language and the Chinese calender system (as seen in their reference to 1844 as the year of the wood dragon). As the Nara Board of Education indicates, they were most likely entirely self taught in mathematics, which is apparent in their use of unique elements in the text and inconsistencies in language.

For example, two circles are labelled *daienkei* 大圓徑 'big circle diameter' and *shōen* 小円 'small circle' respectively on the diagram of the second problem. It is unusual in the *sangaku* tradition that the large circle is labelled 'large circle diameter' on the diagram, since the diameter usually only refers to the width. Different expressions are also used for the term 'circle', as the older variant *en* 圓 is used for the large circle while the modern form *en* 円 describes the smaller. Then in the answer section of problem 2, the character 円 is used for both. This switching between 円 and 圓 does not alter the translation, for the two terms are interchangeable, but nonetheless shows an inconsistency in language. A further example of textual variances appears in the technique sections of problems 1 and 2, where both expressions *jijyō* 自乗 and *jino* 自之 for expressing squaring appear. Although they both reference the same action, usually just one of these terms is used. It may be that such inconsistencies were the result of the calligrapher in charge of painting the tablet making an error. Since this tablet is a replica of an original, it may also be that when the replica was made an error occurred in the copying of the text. These discrepencies in language however point to the author being self taught, for some parts of the labelling system of the mathematical tradition are used correctly while some are used differently, suggesting the author had not fully grasped the standard notation used by mathematicians.

Further evidence of the self taught nature of the author is their use of formulas and rates from the *Jinkōki*, which was the most popular, common, and easy to read mathematical text of the Edo period. The use of this text explains the application of area calculations to problems 1 and 2. If the *Jinkōki* was all the author had access to, then these were the only formulas available to them to work with, for this is what was available in the text. As well as this, there is also an absence of fractions, which were able to be expressed through use of the terms 'dividend' and 'divisor' in *kanbun*. In problem 2 instead of the fraction $\frac{3}{4}$ appearing, the author instead uses the numerical value 75. Since the *Jinkōki* mainly used numerical approximations and expressions, the absense of fractions in the text of this *sangaku* might be explained by a reliance on the *Jinkōki* to learn mathematics, which was the most available mathematical book of the age. Lastly, the fact that a school and master name is not provided on the tablet indicates the author was probably not affiliated with a mathematicial school.

2.8.7 Atago Sangaku

FIGURE 2.19: The *sangaku* at Atago shrine, Fukushima prefecture. (H. Kotera[8]).

Atago shrine 愛宕神社 is located the city of Nihonmatsu in Fukushima prefecture. It enshrines the god *Atago-gongen*, a deity of protection and fire [6, p. 122]. The *sangaku* of this shrine was discovered by accident while the shrine was undergoing renovations in 1994[9]. The exact date it was created remains unknown.

2.8.7.1 Sources and Transcription

The photograph used to transcribe this problem was sourced from H. Kotera's *wasan* website. The original photograph used can be found at:

[8] See http://www.wasan.earth.linkclub.com/fukusima/atago.html

[9] http://www.wasan.earth.linkclub.com/fukusima/atago.html

http://www.wasan.earth.linkclub.com/fukusima/atago.html.

A larger version of this image is located in D.2 in Appendix D. To date no previous transcriptions have been made of this problem.

2.8.7.2 Accompanying Text

AFTER PROBLEM

石森邑	Ishimori District
佐久間治即太即賛編	Saku Maji, Praise literary work

2.8.7.3 Atago Problem

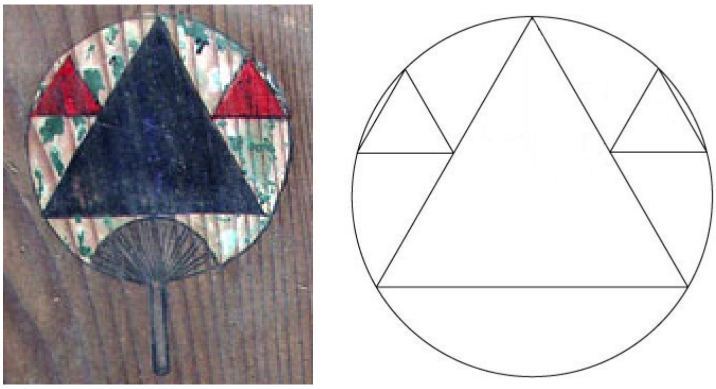


FIGURE 2.20: Left: Atago problem. Right: Transcription. (H. Kotera[10]).

Translation

TRANSCRIPTION	TRANSLATION
今有如図團扇[11]内[illegible]大小三角面只云小三角面一寸問[12]大三角面幾何	As in the diagram, there is a fan which contains a large triangle and small triangles. Say the side length of the small triangles is 1 *sun*. Problem - what is the side length of the large triangle?
答[13]日　大三角面三寸二分三里	Answer: The side length of the large triangle is 3 *sun* 2 *bu* 3 *rin*
術日　置五個開[14]平方加一個乘小角得大三角面合問	Technique: Put 5 *ko* and take the square root. Add 1 *ko* and multiply by the small triangle. Obtain the large triangle side length as required.

[10]See http://www.wasan.earth.linkclub.com/fukusima/atago.html

Translation Notes

The text contains the character  which could not be transcribed and translated. This is because the character appears no longer used in the Japanese language.

It may however be a variation of *yō* 容, which commonly appears after the character *uchi* 内 on other *sangaku*. Another feature of this tablet is the exclusion of the creation date and the characters for dedication *hōnō* 奉納. However in the text the character *san* 賛 for praise does appear. It is unclear whether the name of the author is given in the introductory text, but the district of the author is given as Ishimori, which is close to the shrine location in Fukushima prefecture.

Technical Analysis

The problem asks the observer to find the side length of the large triangle given the side length of the smaller in the fan.

Rendering 1 - Procedure

Problem Say small triangle sides are 1 *sun*. What are the sides of the large triangle?

Answer Large triangle side = 3.23 *sun*

Solution $\sqrt{5}$

$\sqrt{5}+1$

$(\sqrt{5}+1) \times$ small triangle side = large triange side

Large triange side = 3.23 *sun*

Rendering 2 - Formula

Let the sides of the small triangles = s_1, and the sides of the large triangle = s_2.

Problem Say $s_1 = 1$. What is s_2?

Answer $s_2 = 3.23$

Solution $s_2 = (\sqrt{5}+1)s_1$

[11] The characters 團扇 are an older variant of *uchiwa* 団扇, a type of Japanese fan [1, p. 64].
[12] The character *mon* 問 on the tablet is written using Japanese cursive script for the radical 門 [82, p. 336].
[13] This character also uses cursive script for the radical 門 [82, p. 336].
[14] The character *kotae* 答 on the tablet is written using Japanese cursive script [82, p. 270].

Modern Analysis

In Figure 2.21, let A represent the centre of the circular fan, AC be the radius R, s_1 the side length of the small triangles, and s_2 the side length of the large triangle. Assume a right angle triangle $\triangle ABC$ connecting the centre A with the circumference and s_1.

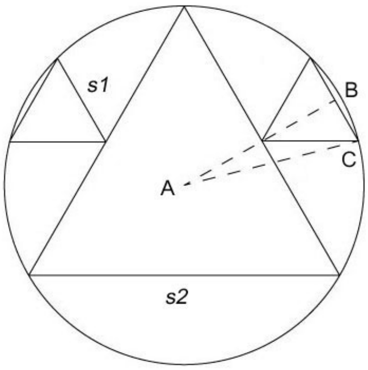


FIGURE 2.21: Atago: Problem analysis

In the triangle $\triangle ABC$

$$BC = \frac{s_1}{2}$$
$$AB = \frac{R}{2} + \frac{\sqrt{3}}{2}s_1$$

Applying the Pythagorean theroem

$$\begin{aligned}
R^2 &= \left(\frac{R}{2} + \frac{\sqrt{3}}{2} \cdot s_1\right)^2 + \left(\frac{s_1}{2}\right)^2 \\
R^2 &= \frac{R}{2}\left(\frac{R}{2} + \frac{\sqrt{3}s_1}{2}\right) + \frac{\sqrt{3}s_1}{2}\left(\frac{R}{2} + \frac{\sqrt{3}s_1}{2}\right) + \frac{{s_1}^2}{4} \\
R^2 &= \frac{R^2}{4} + \frac{2\sqrt{3}Rs_1}{4} + \frac{3{s_1}^2}{4} + \frac{{s_1}^2}{4} \\
R^2 &= \frac{R^2}{4} + \frac{\sqrt{3}Rs_1}{2} + {s_1}^2 \\
\frac{3R^2}{4} &= \frac{\sqrt{3}Rs_1}{2} + {s_1}^2
\end{aligned} \tag{2.6}$$

$$\frac{3R^2}{4} + \frac{1}{2}\left(\frac{\sqrt{3}R}{2}\right)^2 = \left(\frac{\sqrt{3}Rs_1}{2} + {s_1}^2\right) + \frac{1}{2}\left(\frac{\sqrt{3}R}{2}\right)^2$$

$$\frac{15R^2}{16} = \frac{\sqrt{3}Rs_1}{2} + {s_1}^2 + \frac{3r^2}{16}$$

$$\frac{15R^2}{16} = \left(s_1 + \frac{\sqrt{3}R}{4}\right)^2 \tag{2.7}$$

Taking the square root of both sides

$$\sqrt{\frac{15R^2}{16}} = \sqrt{\left(s_1 + \frac{\sqrt{3}R}{4}\right)^2}$$

$$\frac{\sqrt{15R}}{4} = s_1 + \frac{\sqrt{3R}}{4}$$

$$s_1 = \frac{\sqrt{15R}}{4} - \frac{\sqrt{3R}}{4}$$

$$s_1 = \left(\frac{\sqrt{15} - \sqrt{3}}{4}\right) R \tag{2.8}$$

Rearranging to find R in terms of s_1

$$R = \frac{s_1}{\frac{1}{4}(\sqrt{15} - \sqrt{3})}$$

$$R = \frac{4s_1}{\sqrt{3}(\sqrt{5} - 1)}$$

$$R = \frac{4_1}{\sqrt{3}} \times \frac{s_1}{(\sqrt{5} - 1)} \times \frac{\sqrt{5} + 1}{\sqrt{5} + 1}$$

$$R = \frac{4 \times (\sqrt{5} + 1)}{\sqrt{3} \times (\sqrt{5} + 1)} \times \frac{s_1}{\sqrt{5} - 1}$$

$$R = \frac{4(\sqrt{5} + 1)s_1}{\sqrt{3}(5 - 1)}$$

$$R = \frac{(\sqrt{5} + 1)s_1}{\sqrt{3}} \tag{2.9}$$

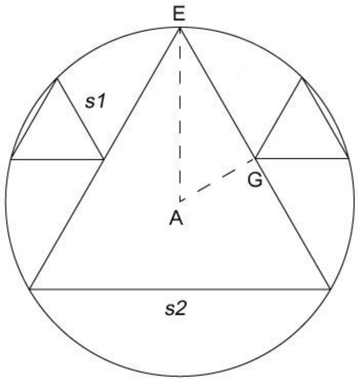


FIGURE 2.22: Atago: Problem analysis cont.

In Figure 2.22 assume a right angle triangle $\triangle AEG$ where

$$EA^2 - AG^2 = EG^2$$
$$R^2 - \left(\frac{R}{2}\right)^2 = \left(\frac{s_2}{2}\right)^2$$
$$\frac{3R^2}{4} = \frac{{s_2}^2}{4}$$
$$R = \pm\frac{s_2}{\sqrt{3}} \tag{2.10}$$

Since the value on the tablet is positive, the solution of $R = \frac{s_2}{\sqrt{3}}$ is adopted. Solving for s_2 gives

$$s_2 = R\sqrt{3}$$
$$s_2 = \frac{(\sqrt{5}+1)s_1}{\sqrt{3}} \times \sqrt{3}$$
$$s_2 = (\sqrt{5}+1)s_1$$
$$s_2 = 3.23 \tag{2.11}$$

2.8.7.4 Comments

A significant amount of working is required to obtain the formula given by the author. Before the large triangle side s_2 can be calculated, first the radius R must be found in terms of s_1 and s_2. The simplified formula $(\sqrt{5}+1)s_1$ given on the tablet does also not provide any clues regarding how to approach the problem, which appears to involve multiple applications of the Pythagorean theorem and a great deal of symbolic manipulation.

Since mechanical calculation devices were used in the Edo period, to ease manual calculation approximations of square roots and other values were available to mathematicians. For instance, in the *Jinkōki*, the following approximations are given - 14142 for $\sqrt{2}$, 833 for $\frac{\sqrt{3}}{2}$, and 316 for π. It is possible that the author of this tablet may have also used an approximation for $\sqrt{5}$, such as 2.23. This would explain the exact result of 3.23 obtained, as $(2.23 + 1) \times 1$ precisely gives the 3.23 on the tablet.

An additional observation of this tablet is the appearance of the golden ratio. If the bases of the two small triangles are connected by a line to form a chord, the ratio of the base of one of the small triangles to the length of the chord between the bases is equal to the golden ratio proportion, expressed algebraically as $\frac{\sqrt{5}+1}{2}$. This proportion can be used to solve for the large triangle side, as the length of the chord between the bases is equal to half the side length of the large triangle. Thus $(\sqrt{5} + 1) \times$ *small side length* is produced by doubling the golden ratio $\frac{\sqrt{5}+1}{2}$. It is unknown however whether the Japanese had knowledge of the golden ratio and applied it to *sangaku* problems.

This tablet lastly gives an everyday object as the mode of displaying the diagram. In this instance, a circular *uchiwa* fan rather than a circle is used. This is purely ornamental and aesthetic, as an ordinary circle would have sufficed to display the problem. The implications of the use of ornamental features such as this are further discussed in chapter 4.

2.8.8 Yoshifuji Mishima Sangaku

FIGURE 2.23: The *sangaku* at Yoshifuji Mishima shrine, Ehime prefecture. (Image by author).

The Yoshifuji Mishima shrine 吉藤三島神社 is located in Yoshifuji town, Matsuyama city, on Shikoku Island. It enshrines the mountain god *Ōyamatsumi no Kami* and was dedicated in 1880 by Matsuoka Tasaburou.

Matsuoka was born in Yoshifuji on the 20th of July 1855. He was the eldest son of the village headman, and since infancy had taken part in arts and activities such as tea ceremony and flower arrangement [86, p. 45]. He is said to have been a 'high color', stylish person with a reputation for being eccentric due to his immersion in mathematical research and meteorological observations from the top of the local mountain Okuboyama with a weather balloon[15]. Matsuoka was a member of the Yamazaki Kizaemon Shōryū school of mathematics, a branch of the Seki school.

An interesting feature of this *sangaku* is the inclusion of a painted scene showing a mathematician teaching a student how to use the *soroban* abacus. An information plaque at the shrine provided by the Matsuyama Board of Education writes "This mathematics tablet is distinctive in that it was offered to the shrine along with a votive tablet with a prayer for success in mathematics and learning". It is said that because his own son was not interested in mathematical study, Matsuoka created this *sangaku* as a "dedication and prayer for academic improvement of [his] son" [86, p. 45]. According to local legend, after this his son did take up study.

[15] See http://www.wasan.earth.linkclub.com/ehime/matuoka.html

2.8.8.1 Sources and Transcription

The transcription provided for this tablet was produced from photographs taken of the original tablet in April 2012. To date no other transcriptions of this problem are known in the literature. A larger image can be found in D.3 of Appendix D.

2.8.8.2 Accompanying Text

温泉郡山崎昌竜門人	Onsen county, Kizaemon Shōryū school
和気郡吉藤町	Wake county, Yoshifuji town
松岡多三郎	Matsuoka Tasaburou
明治十三年七月	Meiji 13th Year [1880], July

2.8.8.3 Yoshifuji Mishima Problem

Translation

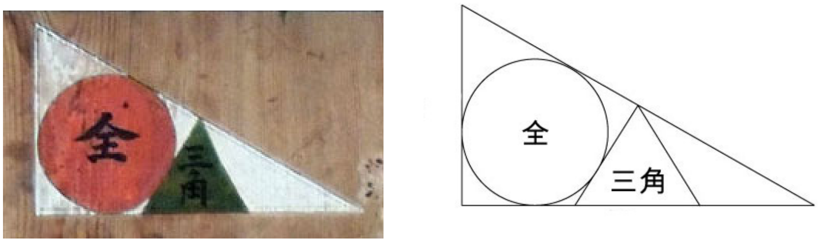


FIGURE 2.24: Left: Yoshifuji Mishima problem. Right: Transcription. (Image by author).

TRANSCRIPTION	TRANSLATION
今有如圖鈎股[16]内容平圓與三角只言鈎壹[17][18]百四拾四間百零八間六合又云股一八合問三角面術幾何	As in the diagram, there is a right angle triangle which contains a circle *hira* (平) and a triangle. Say the short side is 108 *ken* 6 *gō*. As well as this, say that the long side is 144 *ken* 8 *gō*. Problem - what is the side length of the triangle?
答日三角面五十三間有奇	Answer: The side length of the triangle is 53 . . . *ken*

[16]The character *kou* 鈎 is an older variant of 勾 [84, p. 8]. As discussed in section 2.8.5.3, right angle triangles are referred to by the term *koukogen* 勾股弦. However in the tablet text, the labels *kou* 勾 and *ko* 股 appear. I have translated their use in this context to refer to a right angle triangle because these characters are connected to right angle triangles and such a triangle is depicted in the diagram. Literally translated, the opening reads ‘there is a short side and long side which contains a circle and a triangle’. Since these sides are part of the right angle triangle depicted, the author likely intended for their use to refer to such a triangle.

[17]The character *ichi* 壹 is an older variant of the character *ichi* 一for one.

術日置釣自之加股冪平方開之加股名天内減釣餘倍而名地置三個平方開之乘天以釣除之加三個以除地得三角面合問	Technique: Put the short side squared and add the long side squared. Take the square root. Add the long side and name this *heaven*. Inside subtract the short side and double. Furthermore name this *earth*. Put 3 *ko* and take the square root. Multiply *heaven* and divide by the short side. Add 3 *ko* and divide into *earth*. Obtain the triangle side length as required.

Translation Notes

In the answer section, the author includes the characters *yuki* 有奇 after the numerical answer they provide of 53 *ken*. As mentioned in section 2.8.4, the inclusion of *yuki* 有奇 indicates the answer is only an approximation. To show that the author has indicated the value is an approximation, ... has been placed after the translated value of 53.

This tablet uses the labels *heaven* and *earth* as mentioned in section 2.8.5.1 to name series of calculations. As discussed in section 2.8.5.3, right angle triangles are referred to by the term *koukogen* 勾股弦, with the short side labeled *kou* 勾, the long side *ko* 股, and the hypotenuse *gen* 弦. In the tablet text, the labels *kou* 勾 and *ko* 股 appear to describe the triangle the circle and equilateral triangle are inscribed within. I have translated their use in this context to refer to a right angle triangle because these characters, as mentioned in section 2.8.5.3, are connected to right angle triangles and such a triangle is depicted in the diagram. Literally translated, the opening reads 'there is a short side and long side which contains a circle and a triangle'. Since these sides are part of the right angle triangle depicted, the author likely intended for their use to refer to such a triangle.

In this instance, the circle is called *hira* 平 in the tablet text. However, on the diagram, this circle is labelled instead by the character *zen* 全. The character *hira* 平, as discussed in section 2.8.5, translates to "something flat" or "ordinary", and *zen* 全 to "all". Because these characters are intended to operate as nouns and label circles rather than describe them, I have included the original characters and their pronunciation rather than the English translation of the character which may be confused for a description.

Units of measurement which differ to other tablets are also used on this *sangaku*. For instance, the values for the long side and short side of the right angle triangle are 144 *ken* 8 *gō* 一百四拾四間八合 and 108 *ken* 6 *gō* 百零八間六合. Breaking the value down for the short side of the right angle triangle gives *hyaku* 百 - a character attached to numerical values to indicate they are values in the hundreds - first. In this instance, no numerical value is attached, so it reads as just '100'. Interestingly, when 100 is expressed for the long side, the characters *hyaku* 一百 are used instead. *Ichi* 一 means 'one' and *hyaku* 百 'hundred'. In the Japanese tradition, the pronunciation of *ichi* 一 is usually dropped when the two characters appear together, as *hyaku* 百 by itself encompasses the meaning of a single hundred. It is unusual that two different ways of expressing the same value are given, and this may be due to a mistake when making the tablet.

Also, for these values, instead of the measurement terms *shaku* or *sun* - the more common units of lengths - the character *ken* 間 appears. *Ken* 間 is a length measurement which is equivalent to 6 *shaku*. While *shaku* is the traditional standard unit of measurement, Francis D. K. Ching writes

> Another unit of measure, the ken, was introduced in the latter half of Japan's Middle Ages. Although it was originally used simply to designate the interval between two columns and varied in size, the ken was soon standardized for residential architecture. . .The ken, however, was not only a measurement for the construction of buildings. It evolved into an aesthetic module that ordered the structure, materials, and space of Japanese architecture [10, p. 322].

The term came to refer to both a specific measure and an element of architectural design. The use of the measurement *ken*, usually associated with architecture, is uncommon in the Japanese mathematical tradition. Another unusual unit appears after this, being the term *gō* 合. This unit, as discussed earlier in section 2.6, is associated with both areas and volumes. For example, 1 *gō* is the volume of most celebratory *sake* cups. While *gō* can be associated with both areas and volumes, in this instance it is associated with areas and lengths. This is because as seen in the table of area units in section 2.3, 1 *tsubo* = 10 *gō*. 1 *tsubo* is also equal to 1 *ken* squared, so 1 ken^2 = 10 *gō*. This means that *gō* can be connected back to the term *ken* which appears in the text.

But why are these values used instead of the more standard *shaku* system? In the Isaniwa *sangaku* later examined in section 2.8.9, written by the same author, *shaku* and *sun* do appear. This means the author knew how to use this system. Because of this, it was likely then due to personal choice - or the fact that this particular author was known to be a colourful eccentric person - that they decided to use a less conventional system.

Another character that does not appear on the previously examined *sangaku* is *jyu* 拾. This is an alternative written expression of *jyu* 十 '10'. This way of writing '10' is usually restricted in the modern day to legal and financial documents, as a guard against the altering of monetary value (as 一 'one' can be easily be changed to 十 '10' by putting a line through the character) [7, p. 78]. The use of different characters in this problem text once again speaks to the great flexibility in language of *sangaku*. While they usually have the same format, the expressions used for the mathematics is greatly varied between different authors.

Technical Analysis

Using modern symbols, the problem posed by the author and their solution can be expressed in the following manner. A depiction of the diagram with modern labels can be found in Figure 2.25.

Rendering 1 - Procedure

Problem Say the short side of a right angle triangle is 108.6 *ken* and the long side is 144.8 *ken*. What is the side length of the equilaterial triangle?

Answer Triangle side = 53 . . . *ken*

Solution

$$\sqrt{\text{Short side}^2 + \text{long side}^2}$$

$$((\sqrt{\text{Short side}^2 + \text{long side}^2}) + \text{long side}) \to \textit{heaven}$$

$$\textit{Heaven} - \text{short side}$$

$$(\textit{Heaven} - \text{short side}) \times 2 \to \textit{earth}$$

$$\sqrt{3} \times \textit{heaven}$$

$$(\sqrt{3} \times \textit{heaven}) \div \text{short side}$$

$$((\sqrt{3} \times \textit{heaven}) \div \text{short side}) + 3$$

$$\textit{Earth} \div (((\sqrt{3} \times \textit{heaven}) \div \text{short side}) + 3) = \text{triangle side}$$

$$\text{Triangle side} = 53.00047846... \textit{ ken}$$

Rendering 2 - Formula

Let the short side = x, the long side = y, and the side of the equilateral triangle = s the solution can be expressed as an algebraic formula.

Problem Say $x = 108.6$ and $y = 144.8$. What is s?

Answer $s = 53 \ldots$

Solution

$$h = \textit{heaven} = \sqrt{x^2 + y^2} + y$$

$$e = \textit{earth} = 2(\textit{heaven} - x)$$

$$s = \frac{e}{\frac{\sqrt{3} \cdot h}{x} + 3}$$

Modern Analysis

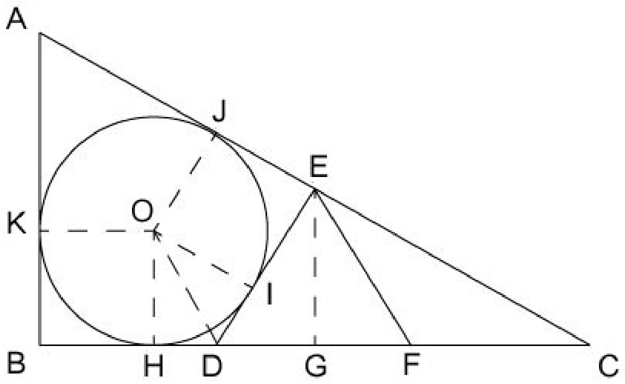


Figure 2.25: Yoshifuji Mishima: Problem analysis

As in Figure 2.25, let $\triangle ABC$ represent the right angle triangle, where AC is the hypotenuse, AB is the short side, and BC is the long side. Say $\triangle DEF$ respresents the equilateral triangle, and O the centre of the circle. Assume the inscribed lines OK, OJ, and OH representing the radius of the circle, and EG as the altitude of $\triangle DEF$. Assume a second right angle triangle $\triangle CEG$ with hypotenuse CE, short side EG, and long side CG.

In $\triangle ABC$

$$\begin{aligned} AB &= AK + KB \\ AC &= AJ + JE + CE \\ BC &= BH + HD + DC \end{aligned} \tag{2.12}$$

Using AB and BC, the hypotenuse AC can be found by applying the Pythagorean theorem

$$AC = \sqrt{AB^2 + BC^2} \tag{2.13}$$

Then the long side BC is added and the short side AB subtracted. This produces (AC + BC) - AB. In $\triangle ABC$

$$\begin{aligned} AB &= AK + BK \\ AK &= AJ \\ BK &= BH \end{aligned} \tag{2.14}$$

Because $AJ + BH$ are equal to $BK + AK$, and $BK + AK = AB$,

$$(AC + BC) - AB = JC + HC \tag{2.15}$$

From Figure 2.25

$$\begin{aligned} HD &= DI \\ JE &= IE \\ DE &= DI + IE \\ &= HD + JE \end{aligned} \tag{2.16}$$

From (2.16), $JC+HC$ can be expressed in terms of DE by substituting DE for $JE+HD$

$$\begin{aligned} JC + HC &= (JE + EC) + (HD + DC) \\ &= DE + CE + DC \end{aligned} \tag{2.17}$$

Now $DC = DG + GC$ and $DG = \frac{1}{2} \cdot DE$, meaning $DE + CE + DC$ can be further expressed in terms of DE as follows

$$\begin{aligned} DE + CE + DC &= DE + CE + (DG + GC) \\ &= DE + CE + \frac{1}{2} \cdot DE + GC \end{aligned} \tag{2.18}$$

When examining the triangle $\triangle ABC$, it can be seen that it is similar to $\triangle EGC$. Given this, the two triangles have similar ratios. A similar ratio expresses in

$$\frac{AC + BC}{AB} = \frac{EC + GC}{EG} \tag{2.19}$$

The value of $(AC + BC) \div AB$ is equal to the long side of $\triangle ABC$ and the value of *earth* $\div$ 2 on the tablet. Since EG is the altitude of the equilateral triangle, and the formula for altitude is $\frac{\sqrt{3}}{2} \times$ *side length*, EG can be expressed in terms of the side length DE as follows

$$\begin{aligned} \frac{EC + GC}{EG} &= \frac{EC + GC}{\frac{\sqrt{3}}{2} \cdot DE} \\ \frac{AC + BC}{AB} &= \frac{EC + GC}{\frac{\sqrt{3}}{2} \cdot DE} \end{aligned} \tag{2.20}$$

Since the triangles $\triangle ABC$ and $\triangle ECG$ are similar, and $\frac{\sqrt{3}}{2} \cdot DE$ forms the altitude of an equilateral triangle DEF with midpoint G

$$\sqrt{3}\left(\frac{AC + BC}{AB}\right) = \frac{EC + GC}{\frac{1}{2} \cdot DE} \tag{2.21}$$

Since the values given are for the short side and long side (from which the hypotenuse can be determined from), the final equation needs to be expressed in terms of these values. Since $AC + BC - AB = (EG + GC) + DE + \frac{1}{2} \cdot DE$, adding 3 to both sides allows $\frac{EG+GC}{\frac{1}{2} \cdot DE}$ to be expressed in terms of $AC + BC - AB$

$$\begin{aligned}
\sqrt{3}\left(\frac{AC + BC}{AB}\right) + 3 &= \frac{EC + GC}{\frac{1}{2} \cdot DE} + 3 \\
&= \frac{EC + GC + \frac{3}{2} \cdot DE}{\frac{1}{2} \cdot DE} \\
&= \frac{2\left(EC + GC + ED + \frac{1}{2} \cdot ED\right)}{DE} \\
&= \frac{AC + BC - AB}{DE}
\end{aligned} \tag{2.22}$$

So that

$$DE = \frac{AC + BC - AB}{\sqrt{3}\left(\frac{AC + BC}{AB}\right) + 3} \tag{2.23}$$

On the tablet we are told $AC + BC - AB = \textit{earth}$, and $\frac{AC+BC}{AB} = \textit{heaven}$. The above method can then be seen to produce the answer on the tablet as follows, where AB is the short side expressed as x in the second rendering

$$DE = \frac{e}{\frac{\sqrt{3} \cdot h}{AB} + 3} \tag{2.24}$$

2.8.9 Isaniwa Sangaku

Figure 2.26: The *sangaku* at Isaniwa shrine, Ehime prefecture. (Image courtesy of the Matsuyama Wasan Society).

The Isaniwa shrine 伊佐爾波神社 is located in the Dogo onsen area of Matsuyama city, on Shikoku island. It enshrines the god *Hachiman*, a deity of war and protection. There are twenty-two *sangaku* in this shrine[18], making it the largest collection of tablets in any one shrine or temple in Japan. The Matsuyama area was highly active in mathematics during the Edo and early Meiji periods, and the majority of mathematicians producing tablets were members of branches of the Seki school. In this section, one tablet from this shrine is examined. This particular *sangaku* was created by Matsuoka Tasaburou - the same author of the *Yoshifuji Mishima* tablet (section 2.8.8) - and placed in the shrine in 1879.

2.8.9.1 Sources and Transcription

This problem has been transcribed but not solved on page 45 of the text *Dōgō Hachiman-gu - Sangaku of Isaniwa Shrine* 道後八幡 - 伊佐爾波神社の算額. Due to the poor quality of the original tablet, I have adopted the transcription found within 道後八幡 - 伊佐爾波神社の算額.

[18]Photographs and transcriptions in Japanese of all twenty-two tablets have been published by the Matsuyama Wasan Society in 道後八幡 - 伊佐爾波神社の算額 [86].

A large photograph of the tablet courtesy of the Matsuyama Wasan Society can be found in D.4 of Appendix D. A copy of the transcription provided in 道後八幡 - 伊佐爾波神社の算額 can also be found in D.5.

2.8.9.2 Accompanying Text

Before problem

奉献	Dedication

After problem

山崎昌竜門人	Yamazaki Masashi Ryū School
和気郡吉藤町	Wake county, Yoshifuji town
松岡多三郎	Matsuoka Tasaburou
秋八月	Fall, August

2.8.9.3 Isaniwa Problem

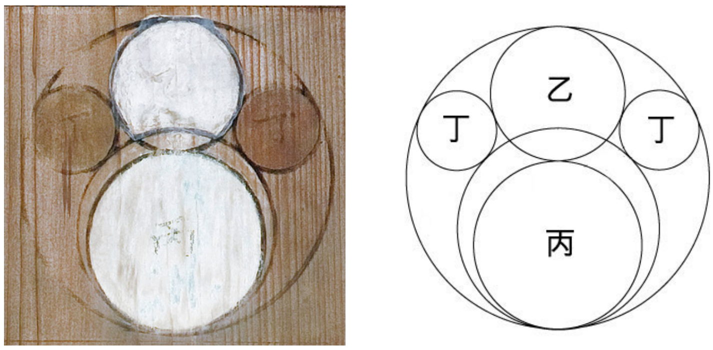


Figure 2.27: Left: Isaniwa problem. Right: Transcription.

Translation

Transcription[19]	Translation
今有如圖平圓内容甲 乙丙丁五圓只云 甲徑一百一十四寸 乙徑七十六寸丙徑 九十五寸問丁徑幾 何乃問術共圓字省之	As in the diagram, inside the circle *hira* 平 are five circles *kō* 甲, *otsu* 乙, *hei* 丙, and *tei* 丁. The diameter of circle *kō* 甲 is 114 *sun*. The diameter of circle *otsu* 乙 is 76 *sun*. The diameter of circle *hei* 丙 is 95 *sun*. Problem - what is the diameter of circle *tei* 丁. As well as this, what is the technique using all circle characters?

答日丁徑四十五寸	Answer: The diameter of circle *tei* 丁 is 45 *sun*
術日相併[20]乙徑丙徑名極内減甲徑餘乘極及丙徑爲実列極自之内減甲徑因[21]乙徑餘以除實得丁徑合問	Technique: Combine the diameter of circle *otsu* 乙 and the diameter of circle *hei* 丙. Name this the *pole*. From the result subtract the diameter of circle *kō* 甲. Multiply the remainder by the *pole* as well as by the diameter of circle *hei* 丙. Make this the dividend. Square the *pole*. From the result subtract the diameter of circle *kō* 甲 multiplied by the diameter of circle *otsu* 乙. Divide into the dividend. Obtain the diameter of circle *tei* 丁 as required.

Translation Notes

The text is presented using *kanbun* without the use of *okurigana* markings as appeared on the *Yoshifuji Mishima* tablet by the same author, and contains some different terminology.

One feature which diverges from others in the tradition is the inclusion of an additional line in the problem section which asks the observer to find the formula in terms of all the circles whose values were given. This section of text is excluded in the translation into modern Japanese given by historians of the Matsuyama area in *Dōgo Hachiman - Isaniwa Jinjya no Sangaku* 道後八幡- 伊佐爾波神社の算額. Instead they interpret the text as advising to only find the value of the smaller circles [86, p. 45]. Their exclusion of this text indicates it is not essential for finding the solution, as it is implied from the giving of the values for the other circles that they are to be used to find the solution. However it does have a valid explanatory function, and makes explicit that the given values of the other circles are to be used to find the solution.

Technical Analysis

Rendering 1 - Procedure

Problem Say the diameter of circle *kō* 甲 is 114 *sun*, the diameter of circle *otsu* 乙 is 76 *sun*, and the diameter of circle *hei* 丙 is 95 *sun*. What is the diameter of the circle *tei* 丁?

[19] From [86, p. 45]

[20] The characters *aiawa* 相併 have a meaning similar to 'combine', and in this context can be seen as indicating a bringing together or combining of two values. The action has been interpreted as addition.

[21] The character *in* 因 appears here. In their translation of the *Tetsujutsu Sankei*, Morimoto and Ogawa translate 因乘の法 as the 'rule of multiplication' [54, p. 255], meaning the character *in* 因 is associated with the concept of multiplication in a mathematical context. With regard to this particular problem, interpreting the character as the action of multiplication (synonymous with *jyō* 乘) fits with the formula produced when solving the problem using a modern approach.

Answer Circle *tei* 丁 $= 45$ *sun*

Solution *otsu* 乙 + *hei* 丙

otsu 乙 + *hei* 丙 → *pole*

Pole − *kō* 甲

(*Pole* − *kō* 甲) × *pole*

((*Pole* − *kō* 甲) × *pole*) × *hei* 丙

Pole^2

Pole^2 − (*kō* 甲 × *otsu* 乙)

(((*Pole* − *kō* 甲) × *pole*) × *hei* 丙) ÷ (*pole*^{2} − (*kō* 甲 × *otsu* 乙)) = *tei* 丁

tei 丁 $= 45$ *sun*

RENDERING 2 - FORMULA

Let the diameter of *kō* 甲 $= a$, the diameter of *otsu* 乙 $= b$, the diameter of *hei* 丙 $= c$, and the diameter of *tei* $= x$.

Problem Say $a = 114$, $b = 76$, and $c = 95$. What is x?

Answer $x = 45$

Solution $p = pole = b + c$

$$x = \frac{c(b+c)((b+c)-a)}{(b+c)^2 - ab} \qquad = \frac{c(p)(p-a)}{p^2 - ab}$$

Modern Analysis

In Figure 2.28, let a be the circle *hira* 平, b be the circle *kō* 甲, c be the circle *otsu* 乙, d be circle *hei* 丙, and e be circles *tei* 丁. Let O represent the centre of the circle a and R its radius. Let r_1 be the radius of circle *kō* 甲, r_2 the radius of circle *otsu* 乙, r_3 the radius of circle *hei* 丙, and r_4 the radius of circles *tei* 丁. Also, assume a triangle $\triangle ABC$ consisting of two separate triangles $\triangle OBC$ and $\triangle OAC$. As in the right-hand diagram of Figure 2.28, in $\triangle OBC$ and $\triangle OAC$

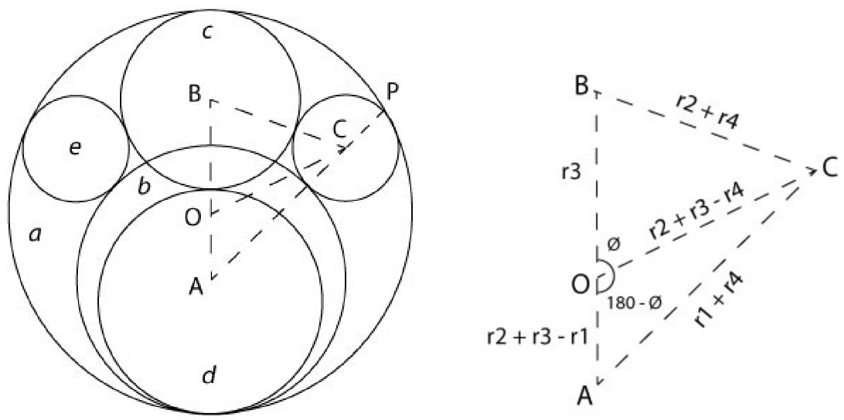


FIGURE 2.28: Isaniwa: Problem analysis

$$BC = r_2 + r_4$$
$$AC = r_1 + r_4$$
$$OC = r_2 + r_3 - r_4$$
$$OA = r_2 + r_3 - r_1$$
$$OB = (r_2 + r_3) - r_2 = r_3$$

Applying the cosine rule ($c^2 = a^2 + b^2 - 2ab\cos\theta$) to $\triangle OBC$

$$BC^2 = OB^2 + OC^2 - 2(OB)(OC)\cos\theta$$
$$(r_2 + r_4)^2 = {r_3}^2 + (r_2 + r_3 - r_4)^2 - 2r_3(r_2 + r_3 - r_4)\cos\theta \tag{2.25}$$

Then applying the cosine rule to triangle $\triangle OAC$

$$\begin{aligned} AC^2 &= OA^2 + OC^2 - 2(OA)(OC)\cos\theta \\ (r_1 + r_4)^2 &= (r_2 + r_3 - r_1)^2 + (r_2 + r_3 - r_4)^2 - 2(r_2 + r_3 - r_1)(r_2 + r_3 - r_4)\cos(180 - \theta) \\ &= (r_2 + r_3 - r_1)^2 + (r_2 + r_3 - r_4)^2 - 2(r_2 + r_3 - r_1)(r_2 + r_3 - r_4)(-\cos\theta) \\ &= (r_2 + r_3 - r_1)^2 + (r_2 + r_3 - r_4)^2 + 2(r_2 + r_3 - r_1)(r_2 + r_3 - r_4)\cos\theta \end{aligned} \tag{2.26}$$

Since (2.25) and (2.26) use the same variables and assigned values - such that r_1 in (2.25) $= r_1$ in (2.26), and similarly for r_2, r_3, and r_4 - they can be treated as equations in the same system. Before using elimination to solve this system for r_4, the equation in (2.25) can be multiplied by $(r_2 + r_3 - r_1)$ and the equation (2.26) by r_3 to allow for $\cos\theta$ to be eliminated. The first multiplication on (2.25) gives

$$\begin{aligned}
&(r_2 + r_4)^2(r_2 + r_3 - r_1) \\
&\qquad = {r_3}^2(r_2 + r_3 - r_1) + (r_2 + r_3 - r_4)^2(r_2 + r_3 - r_1) \\
&\qquad\quad - 2r_3(r_2 + r_3 - r_4)(r_2 + r_3 - r_1)\cos\theta
\end{aligned} \tag{2.27}$$

The second multiplication on (2.26) by r_3 gives

$$\begin{aligned}
&r_3(r_1 + r_4)^2 \\
&\qquad = r_3(r_2 + r_3 - r_1)^2 + r_3(r_2 + r_3 - r_4)^2 + 2r_3(r_2 + r_3 - r_1)(r_2 + r_3 - r_4)\cos\theta
\end{aligned} \tag{2.28}$$

Then, eliminating the $\cos\theta$ in (2.27) and (2.28)

$$\begin{aligned}
&(r_2 + r_4)^2(r_2 + r_3 - r_1) + r_3(r_1 + r_4)^2 \\
&\qquad = {r_3}^2(r_2 + r_3 - r_1) + (r_2 + r_3 - r_4)^2(r_2 + r_3 - r_1) + r_3(r_2 + r_3 - r_1)^2 \\
&\qquad\quad + r_3(r_2 + r_3 - r_4)^2 \\
&\qquad = r_3(r_2 + r_3 - r_1)(r_2 + 2r_3 - r_1) + (r_2 + r_3 - r_4)^2(r_2 + 2r_3 - r_1)
\end{aligned} \tag{2.29}$$

On each side of the equation (2.29) the coefficient of ${r_4}^2$ is equal to $r_2 + 2r_3 - r_1$. Where LHS is the lefthand side of the equation, and RHS is the righthand side of the equation

$$\begin{aligned}
LHS &= (r_2 + r_4)^2(r_2 + r_3 - r_1) + r_3(r_1 + r_4)^2 \\
&= {r_4}^2(r_2 + r_3 - r_1) + {r_4}^2 r_3 + {r_2}^2(r_2 + r_3 - r_1) + 2r_2r_4(r_2 + r_3 - r_1) \\
&\quad + 2r_1r_3r_4 + {r_1}^2 r_3 \\
&= {r_4}^2(r_2 + 2r_3 - r_1) + {r_2}^2(r_2 + r_3 - r_1) + 2r_2r_4(r_2 + r_3 - r_1) + 2r_1r_3r_4 + {r_1}^2 r_3 \\
RHS &= r_3(r_2 + r_3 - r_1)(r_2 + 2r_3 - r_1) + (r_2 + r_3 - r_4)^2(r_2 + 2r_3 - r_1) \\
&= {r_4}^2(r_2 + 2r_3 - r_1) + {r_3}^2(r_2 + 2r_3 - r_1) + {r_2}^2(r_2 + 2r_3 - r_1) \\
&\quad + 2r_2r_3(r_2 + 2r_3 - r_1) - 2r_3r_4(r_2 + 2r_3 - r_1) - 2r_2r_4(r_2 + 2r_3 - r_1) \\
&\quad + r_3(r_2 + r_3 - r_1)(r_2 + 2r_3 - r_1)
\end{aligned} \tag{2.30}$$

Comparing the coeffients for ${r_4}^2$ on each side, the single equation can be rearranged to solve for r_4

$$\begin{aligned}
&{r_2}^2(r_2 + r_3 - r_1) + 2r_2r_4(r_2 + r_3 - r_1) + 2r_1r_3r_4 + {r_1}^2r_3 \\
&\quad = {r_3}^2(r_2 + 2r_3 - r_1) + {r_2}^2(r_2 + 2r_3 - r_1) + 2r_2r_3(r_2 + 2r_3 - r_1) \\
&\qquad - 2r_3r_4(r_2 + 2r_3 - r_1) - 2r_2r_4(r_2 + 2r_3 - r_1) + r_3(r_2 + r_3 - r_1)(r_2 + 2r_3 - r_1) \\
&2r_2r_4(r_2 + r_3 - r_1) + 2r_1r_3r_4 + 2r_3r_4(r_2 + 2r_3 - r_1) + r_2r_4(r_2 + 2r_3 - r_1) \\
&\quad = {r_3}^2(r_2 + 2r_3 - r_1) + {r_2}^2(r_2 + 2r_3 - r_1) + 2r_2r_3(r_2 + 2r_3 - r_1) \\
&\qquad + r_3(r_2 + r_3 - r_1)(r_2 + 2r_3 - r_1) - {r_2}^2(r_2 + r_3 - r_1) - {r_1}^2r_3 \\
&r_4(2r_2(r_2 + r_3 - r_1) + 2r_1r_3 + 2(r_2 + r_3)(r_2 + 2r_3 - r_1)) \\
&\quad = r_3(r_2 + r_3 - r_1)(r_2 + 2r_3 - r_1) + (r_2 + r_3)^2(r_2 + 2r_3 - r_1) - {r_2}^2(r_2 + r_3 - r_1) \\
&\qquad - {r_1}^2r_3 \\
&2r_4(2{r_2}^2 + 4r_2r_3 - 2r_2r_1 + 2{r_3}^2) = 4r_3(r_2 + r_3)(r_2 + r_3 - r_1) \qquad (2.31)
\end{aligned}$$

This can then be rearranged to produce r_4

$$r_4 = \frac{r_3(r_2 + r_3)(r_2 + r_3 - r_1)}{(r_2 + r_3)^2 - r_1r_2} \qquad (2.32)$$

2.8.9.4 Comments

The difference in language between this *sangaku* and the previously examined Yoshifuji by the same author may be explained by the tablets being created for different purposes and audiences. The Yoshifuji tablet was created with the purpose of asking the gods for help to encourage Matsuoka's son to study mathematics. The use of *okurigana* was most likely due to his son being the intended audience, and as being younger and less educated in mathematical language he would be more likely to understand *kanbun* if it had markings to indicate how it was read in Japanese. The Isaniwa tablet however was created with the purpose of being considered a mathematical work on par with others dedicated by mathematicians of the Matsuyama area. It uses different forms of expression and language due to the audience being well studied mathematicians knowledgeable of the terms, and sticks to the same visual format as other tablets which were also dedicated to that particular shrine.

2.8.10 Okiku Inari Sangaku

FIGURE 2.29: The *sangaku* at Okiku Inari shrine, Gunma prefecture. (Image by author).

The Okiku Inari shrine 於菊稲荷神社 is located in Takasaki city, Gunma prefecture. It enshrines the fox god *Inari* who is associated with rice, and contains replicas of two *sangaku* dedicated in the 1820s. In this section, I examine and solve problems from one of these two *sangaku*. The original version of the examined *sangaku* was dedicated to the shrine in 1820 and restored by a local historical group in Shinmachi in May 1978. The first replica was placed in the shrine in 1987, and the second in 2007. According to the oral history of the shrine, the original and first replica were both destroyed by fire and floods, so the latest replica from 2007 is worked from.

The author of the examined *sangaku* was Maruyama Sajūrō 丸山佐十郎平. He was a member of the Seki school of mathematics, and taught by Masuo Sandayu Ryōkyō 増尾三太夫良恭. The second *sangaku* hung in this shrine was by a different author named Ohori Tatsugoro 大堀辰五郎, also part of a different branch of the Seki school [38, p. 251-2]. Ohori's tablet was dedicated in 1826, six years after Maruyama's.

2.8.10.1 Sources and Transcription

The first problem on this tablet has been previously transcribed and solved in the Japanese *Sangaku ni manabu* 算額に学ぶ by Ohara Shigeru [84, p 87]. As well as this all problems have been transcribed in the the text *Gunma no sangaku* 群馬の算額 which contains transcriptions of *sangaku* problems in Gunma prefecture.

These transcriptions as well as photographs of the original tablet taken by myself in 2012 have been used to produce the transcriptions for this section. Unfortunately due to poor lighting my photographs were not of high quality. For this reason a clearer photograph from the Wasan website of H. Kotera has been provided in D.6 of Appendix D.

2.8.10.2 Accompanying Text

奉納　Dedication

AFTER PROBLEMS:

文政三年歳次庚辰五月	Bunsei era 3rd year (1820). Seventeenth year of sexagenary circle. May.
関流七伝増尾三太夫良恭門人 丸山佐十郎平	Seki school 7th Generation - Masuo Sandayu Ryōkyō school Maruyama Sajūrō

2.8.10.3 Okiku Inari: First Problem

Translation

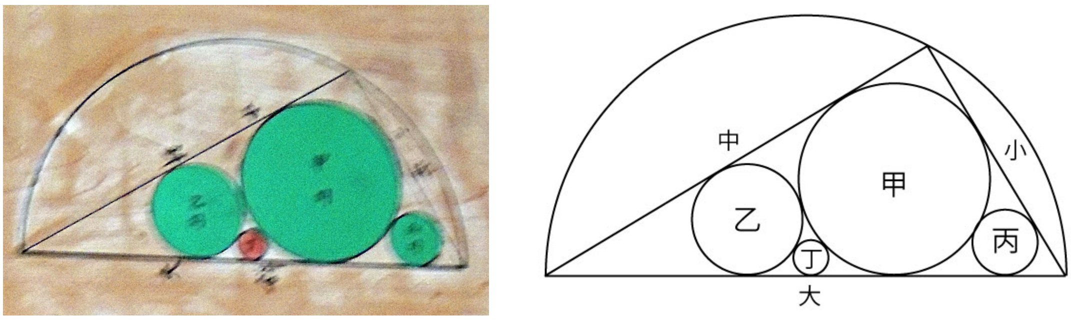


FIGURE 2.30: Left: First Okiku Inari problem. Right: Transcription. (Image by author.)

TRANSCRIPTION	TRANSLATION
今有如圖半圓内作[22]三斜容甲乙丙丁四圓只云大斜若干[23]小斜若干問丁圓径幾何	As in the diagram, there is a semicircle which inside make three lines that contain four circles *kō* 甲, *otsu* 乙, *hei* 丙, and *tei* 丁. Say the large line (大) is known and the small line (小) is known. Problem - what is the diameter of the circle *tei* 丁?
答日　依左術[24]得丁圓径	Answer: See the technique to the left for circle *tei* 丁
術日　別求中斜置併小斜中斜内減大斜余名東[25]以減中斜余乘大斜平方開之以減中斜自之得数以東除之得丁圓径合問	Technique: Obtain the medium line. Combine the small and middle line. Inside subtract the large line. Name the remainder *east*. Subtract from the medium line and multiply the remainder by the large line. Take the square root and subtract from the medium line. Square the result. Divide into *east*. Obtain the diameter of circle *tei* 丁 as required.

Translation Notes

This problem has been previously transcribed and solved in the Japanese *Sangaku ni manabu* 算額に学ぶ by Ohara Shigeru [84, p 87]. However, the transcription of the original tablet text Shigeru provides uses different terminology. He also represents the diagram differently - with the semi-circle missing and the right-angle triangle being flipped so that the long side is the bottom-most line - gives different labels for the sides of the right-angle triangle, and omits the circle *hei* 丙. A copy of the transcription provided by Shigeru on page 87 of 算額に学ぶ is listed below, with sections differing from the original tablet highlighted in yellow.

[22]The character use of the *saku* 作 is unusual here, and can translated as 'to make/produce/build'. The combination 内作 used in the text has an instructive feel to it and can be literally translated as 'inside make'.

[23]The term *jyakkan* 若干 is here used in the place of numerical. As discussed in section 2.8.5.2, *jyakkan* 若干 can be translated as 'is known', and is used to indicate that it is in terms of the figure(s) the term follows that the sought after figure is to be expressed.

[24]Instead of providing a numerical answer, in this instance the reader is advised to refer to the formula in the technique section. Refer to section 2.8.5.2 for more details.

[26]Instead of using the more common terms '*heaven*', '*earth*', or '*pole*' to label a set of calculations (see 2.8.5.1), the author chooses '*east*'. This may be due to the four directions of east, west, north, and south being opposites or extremes of one another.

Shigeru's Transcription	Literal Translation
今有如図鉤股内容甲乙丙丁円鉤若干弦若干問求丁円径其術如何	As in the diagram, there is a right-angle which contains circles *kō* 甲, *otsu* 乙, and *tei* 丁. Say the short side is known and the hypotenuse is known. Problem - obtain the method for the diameter of the circle *tei* 丁.
答曰　依左術得丁円径	Answer: See the technique to the left for circle *tei* 丁
術曰　別求股置併鉤股内減弦余名東以減股余乗弦平方開之以減股余自之得数以東除之得丁円径合問	Technique: Obtain the long side. Combine the short side and the long side. Inside subtract the hypotenuse. Name the remainder *east*. Subtract from the long side and multiply the remainder by the hypotenuse. Take the square root and subtract from the long side. Square the remainder. Divide into *east*. Obtain the diameter of circle *tei* 丁 as required.

In the original tablet text, the sides of the triangle are labelled according to their length with the 'large' *dai* 大, 'medium' *chū* 中, 'small' *shō* 小 system from section 2.8.5. In his transcription of the text however, Shigeru adopts the labels *kou* 勾, *ko* 股, and *gen* 弦 for the sides. Shigeru also uses the more modern forms of the characters *en* 圓 and *zu* 圖.

Another noticeable difference is in the phrasing. The text from the tablet photographed in Figure 2.29 states "Problem - what is the diameter of the circle *tei* 丁?" while Shigeru's text can be translated as "Problem - obtain the method for the diameter of the circle *tei* 丁".

The differences between the original text and that of Shigeru are likely due to the audience of his work being modern Japanese readers. Throughout his work, Shigeru consistently adopts the modern character for circle *en* 円 instead of the character *en* 圓, as well as *zu* 図 for 圖. In the three other problems with right-angle triangles Shigeru transcribes in his text (problems 13, 76, and 78), he also uses more modern terminology *kou* 勾, *ko* 股, and *gen* 弦 for the sides of right-angle triangles. This unfortunately presents a less literal transcription. I have instead used a transcription directly based off the tablet itself which I photographed in April of 2014.

Another transcription which differs only in the exchanging of the modern *en* 円 for *en* 圓 and *zu* 図 for 圖 can be found on page 211 of *Gunma no sangaku* 群馬の算額 which contains transcriptions of *sangaku* found within Gunma prefecture [38, p. 211].

Technical Analysis

Rendering 1 - Procedure

Problem Say the semicircle contains a triangle and four circles *kō* 甲, *otsu* 乙, *hei* 丙, and *tei* 丁. Say the short side and hypotenuse of the triangle are known. What is the diameter of circle *tei* 丁?

Answer See the technique to the left

Solution

$$\begin{aligned}
&\text{Long side} \quad \left(= \sqrt{\text{hypotenuse}^2 - \text{short side}^2}\right) \\
&((\text{Short side} + \text{long side}) - \text{hypotenuse}) \to \textit{east} \\
&\text{Long side} - \textit{east} \\
&(\text{Long side} - \textit{east}) \times \text{hypotenuse} \\
&\text{Long side} - \sqrt{(\text{Long side} - \textit{east}) \times \text{hypotenuse}} \\
&\text{Long side} - \left(\sqrt{(\text{Long side} - \textit{east}) \times \text{hypotenuse}}\right)^2 \\
&\frac{\text{Long side} - \left(\sqrt{(\text{Long side} - \textit{east}) \times \text{hypotenuse}}\right)^2}{\textit{east}}
\end{aligned} \tag{2.33}$$

Rendering 2 - Formula

Let the short side $= a$, the long side $= b$, the hypotenuse $= c$, and the diameter of the circle *tei* 丁 $= x$.

Problem What is x given a and c?

Answer

Solution

$$e = \textit{east} = (a + b) - c$$

$$x = \frac{b - \left(\sqrt{(b-e)\cdot a}\right)^2}{e}$$

Modern Analysis

In the interest of representing the tablet and all its problems in their entirety, an analysis based on Shigeru's is provided in this section. However, as noted, I have adopted

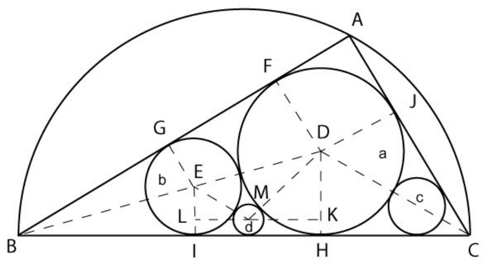


FIGURE 2.31: Okiku Inari: First problem analysis

a transcription which sticks to the tablet text.

As in Figure 2.31, let there be a semicircle which contains a right angle triangle $\triangle ABC$. Say inside the triangle there are four circles a, b, c, and d with respective diameters d_1, d_2, d_3 and d_4. Assume supplementary right angle triangles $\triangle BDH/BDF$ and $\triangle BEI/BEG$ formed by connecting the centres of a and b with the point B of $\triangle ABC$. Assume further lines DF, EG, LI, HK, and triangles $\triangle EML$ and $\triangle DMK$.

In Shigeru's presentation of the problem, the right angle triangle $\triangle ABC$ is positioned differently, rotated 180 degrees counter-clockwise. The circle *tei* 丁 is also placed between the circles a and b on the line segment FG as in Figure 2.31. The triangle is also not placed within a semicircle in the diagram given in this text. However, his method is still applicable.

The solution first focuses on finding the length of the long side AB by applying the Pythagorean theorem. Then the diameter of the largest circle in the triangle is obtained

$$AB^2 = BC^2 - AC^2$$
$$d_1 = (AC + AB) - BC$$

The triangles $\triangle BDF$ and $\triangle BEG$ are similar and in the same ratio to one another. Due to this,

$$DF : EG = BF : BG$$
$$BF = AB - AF$$
$$BG = AB - AF - FG$$
$$DF = AF = \frac{d_1}{2}$$
$$EG = \frac{d_2}{2}$$
$$FG = \sqrt{d_1 d_2} \tag{2.34}$$

From (2.34)

$$
\begin{aligned}
\frac{d_1}{2} : \frac{d_2}{2} &= \left(AB - \frac{d_1}{2}\right) : \left(AB - \frac{d_1}{2} - \sqrt{d_1 d_2}\right) \\
d_1 : d_2 &= (2AB - d_1) : (2AB - d_1 - 2\sqrt{d_1 d_2}) \\
(2AB - d_1)d_2 &= (2AB - d_1)d_1 - 2d_1\sqrt{d_1 d_2} \\
(2AB - d_1)d_2 + 2d_1\sqrt{d_1 d_2} - (1AB - d_1)d_1 &= 0 \qquad (2.35)
\end{aligned}
$$

From (2.35), Shigeru determines the value of $\sqrt{d_2}$ as

$$\sqrt{d_2} = \frac{\sqrt{d_1}(2\sqrt{AB - d_1)BC} - d_1)}{2AB - d_1} \qquad (2.36)$$

Knowing $\sqrt{d_1}$ and $\sqrt{d_2}$, Shigeru indicates that the following formula can be applied to solve for d_4

$$\sqrt{d_4} = \frac{d_1 d_2}{(\sqrt{d_1} + \sqrt{d_2})^2} \qquad (2.37)$$

From (2.36) and (2.37)

$$
\begin{aligned}
\sqrt{d_4} &= \frac{\dfrac{\sqrt{d_1}(2\sqrt{(AB - d_1)BC} - d_1)\sqrt{d_1}}{2AB - d_1}}{d_1 + \dfrac{(2\sqrt{(AB - d)BC} - d_1)\sqrt{d_1}}{2AB - d_1}} \\
d_4 &= \frac{(AB - \sqrt{(AB - d_1) - BC})^2}{d_1} \qquad (2.38)
\end{aligned}
$$

Where $x = d_4$, $e = d_1$, and $b = AB$, this matches the formula given on the tablet of $x = \dfrac{b - \left(\sqrt{(b-e)\cdot a}\right)^2}{e}$.

2.8.10.4 Comments

The slightly different configuration of the diagram in Shigeru's text to that on the *sangaku* is possible because the diagram contains figures which are not required for the solution. For instance, the third circle *hei* 丙 represented by d_3 is not used in the solution and can be positioned differently on the diagram or omitted. Also, the circle d_4 can be located either between d_1 and d_2 on the line segment FG or HI without altering the solution. The diagram can be formed in different ways and have the same solution.

Another feature of this problem is the omission of the calculation of the long side AB of the right angle triangle in the technique section. The author instructs to 'obtain' the long side, but does not provide information on how this is done. This assumes that the reader is versed in the Pythagorean theorem and how to apply it. The including of values that have not been explicitly calculated does not occur in any other problems examined in this thesis. This calculation may have been omitted because the Pythagorean theorem is associated with the very conception of a right angle triangle in the Japanese tradition, as discussed in section 2.8.5.3. So by recognising it was a right angle triangle in the diagram, the observer would already be considering the theorem. However, it may be recalled in the Yoshifuji Mishima tablet (section 2.8.8), a side length of the right angle triangle in that problem was obtained also by application of the Pythagorean theorem, and in that instance the author did explicitly give the calculation required in the technique section.

2.8.10.5 Okiku Inari: Second Problem

Translation

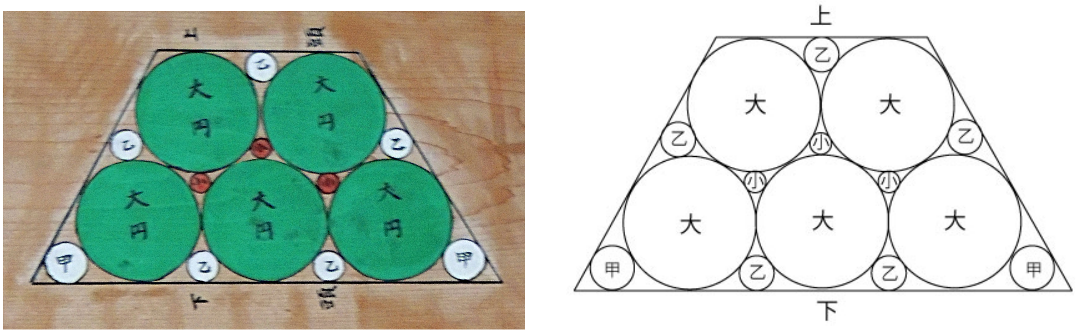


FIGURE 2.32: Left: Second Okiku Inari problem. Right: Transcription. (Image by author.)

TRANSCRIPTION	TRANSLATION
今有如圖梯内容大小甲乙圓 只云上頭若干[27]問 小圓径幾何	As in the diagram, there is a trapezium containing large (大) and small circles (小), and circles *kō* 甲 and *otsu* 乙. Say the diameter of the upper side is known. Problem - what is the diameter of the small circles?
答日　依左術[28]得小圓径	Answer: See the technique to the left

[27]The term *jyakkan* 若干 is used to reference the upper side of the trapezium and informs the observer that while a numerical value is not given for this side, they are to treat it as if it is known, and find the small circle diameter in terms of it.

[28]The answer section directs the observer towards the technique section for the solution.

術日　置六個七分五厘平方開之内減二個五分余乗只云数得小圓径合問	Put 6 *ko* 7 *bu* 5 *rin* and take the square root. Inside subtract 2 *ko* 5 *bu*. Multiply by the value of the upper side. Obtain the small circle diameter as required.

Translation Notes

A transcription of this problem can also be found on page 211 of *Gunma no sangaku* 群馬の算額 [38, p. 211]. As with the previous problem, the modern characters *en* 円 and *zu* 図 replace *en* 圓 and *zu* 圖. I have created my transcription based on a photograph of the tablet and used the previous transcription in *Gunma no sangaku* for guidance.

The sides of the trapezium are labelled unusually in this problem, with the author using *ue-atama* 上頭 for the upper side and *shita-atama* 下頭 for the lower. The characters *ue* 上 and *shita* 下 mean 'upper' and 'lower', and *atama* 頭 translates to 'head'. A more common label for sides of figures is *hasu* 斜, and this label is used for the sides of the trapezium that presents in the third problem on the tablet. In this context however, *ue-atama* 上頭 can be treated as similar to *uehasu* 上斜, and considered as 'upper side' or 'upper line'.

Technical Analysis

Rendering 1 - Procedure

Problem Say the trapezium contains large and small circles, and circles *kō* 甲 and *otsu* 乙. What are the small circles in terms of the upper side of the trapezium?

Answer See the technique to the left

Solution

$6\ ko\ 7\ bu\ 5\ rin$

$\sqrt{6\ ko\ 7\ bu\ 5\ rin}$

$\sqrt{6\ ko\ 7\ bu\ 5\ rin} - 2\ ko\ 5\ bu$

$(\sqrt{6\ ko\ 7\ bu\ 5\ rin} - 2\ ko\ 5\ bu) \times \text{upper side} = \text{small circle}$

Rendering 2 - Formula

Let the small circle diameter $= x$ and the upper side of the trapezium $= a$.

Problem What is x in terms of a?

Answer

Solution $x = (\sqrt{6.75} - 2.5)a$

Modern Analysis

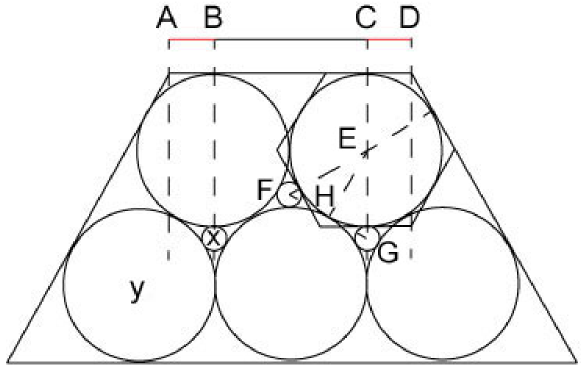


FIGURE 2.33: Okiku Inari: Second problem analysis

As in Figure 2.33, let AD be the the upper side of the trapezium, d_1 the diameter of the large circles, and d_2 the diameter of the small circles nested inbeween the larger. Assume there is a hexagon circumscribing one of the large circles which has a side length s. Assume an equilateral triangle $\triangle EFG$ inside a hexagon with an altitude EH.

First the side length of the upper side AD is found in terms of d_1. In Figure 2.33, AD is represented by the segments AB, BC, and CD. The segment BC is equivalent to d_1. The lengths AB and CD can be found in terms of d_1 by inscribing the circle within a hexagon, as the length of each side of this assumed hexagon s is equal to $2AB$ or $2CD$, such that

$$\begin{aligned}
&AD = AB + BC + CD \\
&BC = d_1 \\
&AD = d_1 + AB + CD \\
&AB = CD = \frac{s}{2} \\
&AD = d_1 + s \qquad (2.39)
\end{aligned}$$

Since the hexagon is made up of 6 equilateral triangles, d_1 can be calculated in terms of s by rearranging the formula for the height of an equilateral triangle

$$\frac{s}{2}\sqrt{3} = \frac{d_1}{2}$$
$$\frac{s\sqrt{3}}{2} = \frac{d_1}{2}$$
$$s\sqrt{3} = d_1$$
$$s = \frac{d_1}{\sqrt{3}}$$
$$AD = d_1 + \frac{d_1}{\sqrt{3}} \tag{2.40}$$

This can be rearranged to put d_1 in terms of AD

$$d_1 = \frac{3AD}{3+\sqrt{3}} \tag{2.41}$$

In triangle $\triangle EFG$, it can been seen that

$$EH = \frac{d_1}{2}$$
$$s = \frac{d_1}{2} + d_2$$

Rearranging to solve for $\frac{d_2}{2}$ and substituting 2.41 for d_1

$$d_2 = s - \frac{d_1}{2}$$
$$\frac{d_2}{2} = \frac{d_1}{\sqrt{3}} - \frac{d_1}{2}$$
$$\frac{d_2}{2} = \frac{\left(\dfrac{3AD}{3+\sqrt{3}}\right)}{\sqrt{3}} - \frac{\left(\dfrac{3AD}{3+\sqrt{3}}\right)}{2}$$

$$\frac{d_2}{2} = \frac{(3\sqrt{3}-5)AD}{4}$$
$$d_2 = \frac{(3\sqrt{3}-5)AD}{2}$$
$$d_2 = \left(\frac{3\sqrt{3}}{2} - 2.5\right) AD = (\sqrt{6.75} - 2.5)AD \tag{2.42}$$

2.8.10.6 Okiku Inari: Third Problem

Translation

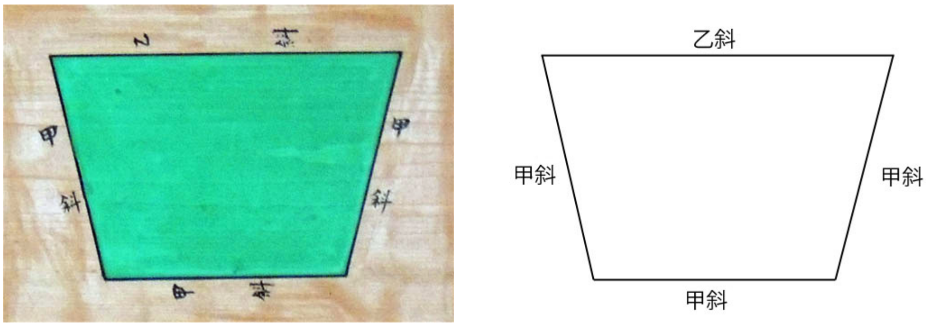


FIGURE 2.34: Left: Third Okiku Inari problem. Right: Transcription. (Image by author).

TRANSCRIPTION	TRANSLATION
今有如圖四斜[29]甲斜容一寸欲積最多[30]問乙斜幾何	As in the diagram, there is a four sided shape. Side *kō* 甲 of 1 *sun* is the most numerous. Problem - what is the side *otsu* 乙?
答日　得乙斜二寸	Obtain side *otsu* 乙 as 2 *sun*
術日　置甲斜倍之得乙斜合問	Technique: Put side *kō* 甲 and double it. Obtain side *otsu* 乙 as required.

[29]The characters *shi hasu* 四斜 'four line' or 'four side' are used instead of the more common character *tei* 梯 to describe the trapezium in the diagram.

[30]Instead of stating how many times the sides *kō* 甲 of 1 *sun* appear, the author writes that this line is the 'most numerous'. This form of description is unusual, for usually a specific number or no reference to number is given.

Translation Notes

This problem is also transcribed in *Gunma no sangaku* 群馬の算額 [38, p. 211] with *zu* 図 replacing *zu* 圖. As with the other two problems, I have worked from a photograph.

Technical Analysis

RENDERING 1 - PROCEDURE

Problem Say the sides *kō* 甲 are 1 *sun*. What is the side *otsu* 乙?

Answer Side *otsu* 乙 = 2 *sun*

Solution Side *kō* 甲 × 2 = Side *otsu* 乙

RENDERING 2 - FORMULA

Let the side *otsu* 乙 $= x$ and side *kō* 甲 $= a$.

Problem Say a is 1. What is x?

Answer $x = 2$

Solution $x = 2a$

Modern Analysis

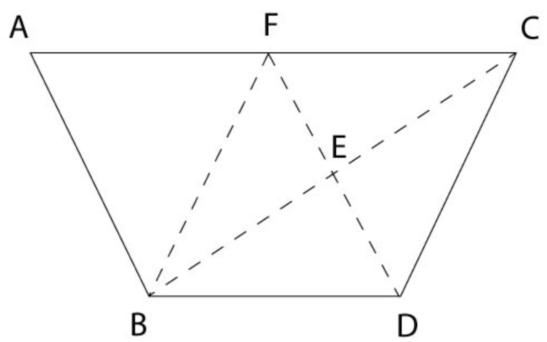


FIGURE 2.35: Okiku Inari: Third problem analysis

This problem is underdetermined as AC could be any length between BD and *3BD*. The problem can be solved when assuming the extra condition that $\angle A$ and $\angle C$ are 60°.

Assuming this, let AC represent the upper side *otsu* 乙, and *AB, BD, DC* represent the three sides *kō* 甲. Assume a triangle $\triangle ABC$ and a line BC which is split into BE and *EC*. Assume the auxiliary line ED produces two triangles $\triangle BDE$ and $\triangle CDE$. Assuming $\angle A$ and $\angle C$ are 60°, the lines BD and CD will be identical in length, and thus the two triangles identical. Together they can form an equilateral triangle, meaning each is a 30-60-90 triangle.

The ratio for 30-60-90 triangles can be applied to determine

$$\begin{aligned}
&BE = \sqrt{3} \times ED \\
&EC = \sqrt{3} \times ED \\
&BC = \sqrt{3} \times 2ED
\end{aligned} \tag{2.43}$$

Knowing the sides BC and AB in $\triangle ABC$, the length AC can be found by applying the Pythagorean theorem

$$\begin{aligned}
&AC^2 = AB^2 + BC^2 \\
&AC^2 = AB^2 + (\sqrt{3} \times 2ED)^2 \\
&AC^2 = 1^2 + (\sqrt{3} \times 1)^2 \\
&AC = 2
\end{aligned} \tag{2.44}$$

2.8.10.7 Comments

This tablet may show a *sangaku* author appealing to a broad audience, for two problems are presented using different language. The third problem noticeably has more basic descriptions (i.e. the trapezium is called a ‘four sided’ figure rather than by the technical term for trapezium as in the first problem). However this problem is also underdetermined, which may be the result of the basic language used.

These three problems also show the different types of problems that present on *sangaku*. There are problems which put figures in terms of another and then use the numerical values given to solve for the numerical value in the answer section, while others contain no numerical values, leaving the observer to rely solely on their own knowledge of geometry to work out the answer. For instance, in the third problem, the very fact that the numerical answer in the answer section is twice the size of the value in the problem section gives the reader some clue about the problem. The reader can look for geometrical principles that might produce this, such as a 30:60:90 triangle with a ratio of 1:2:$\sqrt{3}$. But, in the case of the first two problems, no numerical values were given, so no hints are provided for the reader as to what geometrical principles they might employ.

The non-mathematical text of this tablet also provides some interesting insights. The author includes the characters *Seki ryū* 関流, meaning 'Seki school', to indicate they are a member of a branch of this school, and informs the observer they are of the seventh generation. Also, the author provides the year of dedication both in the traditional Japanese calender (with era name and year of that era) and the Chinese sexagenary cycle. This inclusion of the sexagenary cycle indicates the author was educated not only in mathematics but also other areas of learning such as calender science.

2.8.11 Namigura Inari Sangaku

FIGURE 2.36: The *sangaku* at Namigura Inari shrine, Fukushima prefecture. (Fukushima University[31]).

The Namigura Inari shrine 波倉稲荷神社 is located in Naraha town of Fukushima prefecture. It is of the *Inari* variety, and contains one *sangaku* dedicated in 1921. According to local tradition, this tablet was the result of the study of nine Namigura district locals who were students of the native Japanese mathematics teacher Sekine Kumakichi from the town of Tomioka[32]. The students wrestled with a difficult problem, and the answer they obtained was made into this *sangaku*.

2.8.11.1 Sources and Transcription

This tablet has not been previously transcribed and solved to date in the literature. The transcription I have provided has been derived from examination of the photograph provided by H. Kotera which is found at the following site: http://www.wasan.earth.linkclub.com/fukusi

A larger photograph from Fukushima University can be found in D.7 of Appendix D.

[31]See
[32]See http://is2.sss.fukushima-u.ac.jp/fks-db//txt/10088.002/html/00019.html

2.8.11.2 Accompanying Text

Before problem

奉献	Dedication
蘢田村大字波倉	Namigura Village

After problem

関根熊吉門人	Sekine Kumakichi School
大正拾年貳月朸午	Taisho era, 10th year, 2nd month

2.8.11.3 Namigura Inari Problem

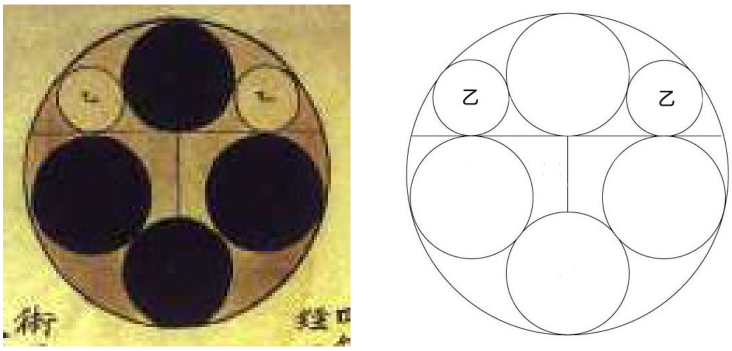


Figure 2.37: Left: Namigura Inari problem. Right: Transcription. (H. Kotera[33]).

Translation

Transcription	Translation
今有如圖大円内容甲円 四個乙円弐個只云大円 径一十七寸問乙円径幾何	As in the diagram, there is a large circle (大) which contains four circles *kō* 甲 and two circles *otsu* 乙. Say the diameter of the large circle is 17 *sun*. Problem - what is the diameter of the circles *otsu* 乙?
答　乙四寸	Answer: The diameter of the circles *otsu* 乙 is 4 *sun*
術日置大円径四倍之以 大径除之得乙円径合問	Technique: Put the large circle diameter and multiply by four. Divide this by the large circle diameter. Obtain diameter of circle *otsu* 乙 as required.

[33]See http://www.wasan.earth.linkclub.com/fukusima/hakura.html

Translation Notes

This tablet uses the character *ni* 弐 for the number two. This is an older variant of the more common *ni* 二 - meaning 2 - and both have an identical meaning.

Technical Analysis

RENDERING 1 - PROCEDURE

Problem Say the diameter of the large circle is 17 *sun*. What is the diameter of the circle *otsu* 乙?

Answer Circle *otsu* 乙 = 4 *sun*

Solution Large circle × 4

(Large circle × 4) ÷ large circle = *otsu* 乙

otsu 乙 = 4 *sun*

RENDERING 2 - FORMULA

Let the diameter the large circle = a, and the diameter of circle *kō* 甲 = x.

Problem Say $a = 17$. What is x?

Answer $x = 4$

Solution $x = \dfrac{4a}{a}$

Modern Analysis

This problem first requires the diameter of the circles *kō* 甲 - labelled b in Figure 2.38 - to be obtained. Then using their diameter and that of the large circle, the value of *otsu* 乙 - labelled x - can be found.

As in Figure 2.38, say the radius of the large circle is R, the radius of the circles b is r_1, and the radius of the circles x is r_2. Assume auxiliary lines AE and FC, and a triangle

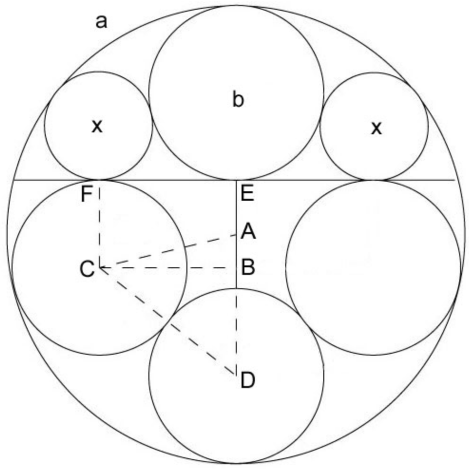


Figure 2.38: Namigura Inari: Problem analysis

$\triangle ACD$ which contains two triangles $\triangle ABC$ and $\triangle BCD$. From Figure 2.38

$$AE = d$$
$$AB = r_1 - d$$
$$AC = (R - r_1)$$
$$BE = FC = r_1$$
$$CD = 2r_1$$
$$BD = R - (r_1 - d) - r_1$$
$$R = 2r_1 + d$$

To first find r_1, the Pythagorean theorem can be applied to $\triangle ABC$ and $\triangle BCD$

$$AC^2 - AB^2 = BC^2$$
$$(R - r_1)^2 - (r_1 - d)^2 = BC^2$$
$$CD^2 - BD^2 = BC^2$$
$$(2r_1)^2 - (R - 2r_1 + d)^2 = BC^2$$
$$(R - r_1)^2 - (r_1 - d)^2 = 4{r_1}^2 - (R - 2r_1 + d)^2 \tag{2.45}$$

Then (2.45) can be rearranged to solve for r_1

$$(R - r_1)^2 - (r_1 - d)^2 = 4{r_1}^2 - (R - 2r_1 + d)^2$$
$$2R^2 + R(-6r_1 + 2d) - 2r_1 d = 0 \tag{2.46}$$

To eliminate the term d, since $R = 2r_1 + d$ it can be determined that $d = R - 2r_1$. Substituting $R - 2r_1$ for d gives

$$2R^2 + R(-6r_1 + 2R - 4r_1) - 2r_1(R - 2r_1) = 0$$
$$4R^2 - 12Rr_1 + 4{r_1}^2 = 0$$
$$R^2 - 3Rr_1 + {r_1}^2 = 0$$

$${r_1}^2 - 3Rr_1 + \frac{9R^2}{4} = \frac{5R^2}{4}$$
$$\left(r_1 - \frac{3R}{2}\right)^2 = \frac{5R^2}{4}$$
$$r_1 = \frac{3R}{2} + \frac{\sqrt{5}}{2} \quad \text{or} \quad r_1 = \frac{3R}{2} - \frac{\sqrt{5}R}{2} \tag{2.47}$$

The calculation $\frac{3R}{2} + \frac{\sqrt{5}}{2}$ gives an approximate value of 13.868 for r_1. Since this is larger than R, the second calculation $\frac{3R}{2} - \frac{\sqrt{5}R}{2}$ is used

$$\begin{aligned} r_1 &= \frac{3R}{2} - \frac{\sqrt{5}R}{2} \\ &= \frac{3 - \sqrt{5}}{2} \cdot R \end{aligned} \tag{2.48}$$

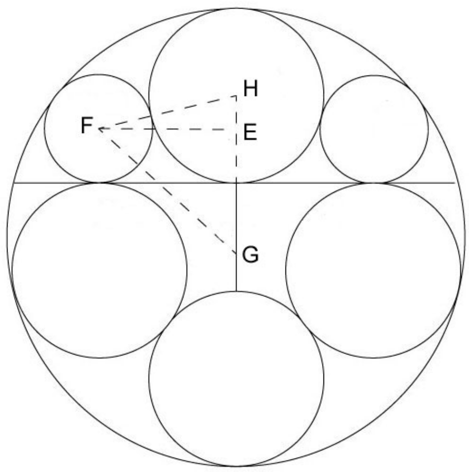


FIGURE 2.39: Namigura Inari: Problem analysis cont.

Having found the radius of r, a right angle triangle $\triangle EFG$ can be assumed as in Figure 2.39 where

$$
\begin{aligned}
&EG = R - 2r_1 + r_2 \\
&FG = R - r_2 \\
&FH = r_1 + r_2 \\
&FE = 2\sqrt{r_1 r_2} \\
&GE = R - 2r_1 + r_2
\end{aligned}
$$

Applying the Pythagorean theorem to solve for the radius r_2 of the small circles

$$
\begin{aligned}
&FE^2 + GE^2 = FG^2 \\
&(2\sqrt{r_1 r_2})^2 + (R - 2r_1 + r_2)^2 = (R - r_2)^2 \\
&4{r_1}^2 - 4Rr_1 + 2Rr_2 = -2Rr_2 \\
&Rr_2 = Rr_1 - {r_1}^2 = r_1(R - r_1) \\
&r_2 = \frac{r_1(R - r_1)}{R}
\end{aligned}
$$

Substituting the previously obtained value of $\frac{3-\sqrt{5}}{2} \cdot R$ for r_1

$$
\begin{aligned}
r_2 &= \frac{\left(\left(\frac{3-\sqrt{5}}{2}\right) R\right)\left(R - \left(\left(\frac{3-\sqrt{5}}{2}\right) R\right)\right)}{R} \\
&= \frac{(3-\sqrt{5})\left(R - \left(\left(\frac{3-\sqrt{5}}{2}\right) R\right)\right)}{2} \\
&= \frac{(3-\sqrt{5})\left(\frac{2R + (\sqrt{5}-3)R}{2}\right)}{2} \\
&= \frac{(3-\sqrt{5})(\sqrt{5}-1)R}{4} \\
&= \frac{(4(\sqrt{5}-8))R}{4} \\
&= \frac{(4(\sqrt{5}-2))R}{4} \\
&= (\sqrt{5}-2)R
\end{aligned}
$$

$$2r_2 = 4.013155617 \tag{2.49}$$

2.8.11.4 Comments

With this problem, the answer produced in the modern analysis does not match that given on the tablet, being 4. There may be a few reasons why this answer and that produced in the modern analysis differ. One explanation is that an approximation has been given, since 4.013 is close to 4. Another reason could be that when calculating on the *soroban* the authors worked out 4.0 and did not calculate further figures.

The formula given in the technique section also differs to what the modern analysis provides, with $r_2 = \frac{4R}{R}$ on the tablet and $r_2 = (\sqrt{5} - 2)R$ in the modern analysis. An interesting feature of the technique is that produces a value of 4 regardless of the other values in the problem. If the radius of the large circle was scaled up to become 20 rather than 17, the answer produced would still be 4. However the formula from the modern analysis would put the small circles at 4.721 when the large circle is 20. Given this, it appears the authors produced a formula for only this specific case instead of a general formula. They may have also predetermined the values in the diagram, setting the small circle value beforehand to be 4 and then working how to obtain it from the larger. It is also the case however that the value of $\sqrt{5} - 2$ can be expressed approximately as 0.23606, and this is similar to the value of $\frac{4}{17}$ which is 0.23529. The authors may have noticed this and incorporated it to obtain their formula.

2.8.12 Kumakabuto Arakashihiko Sangaku

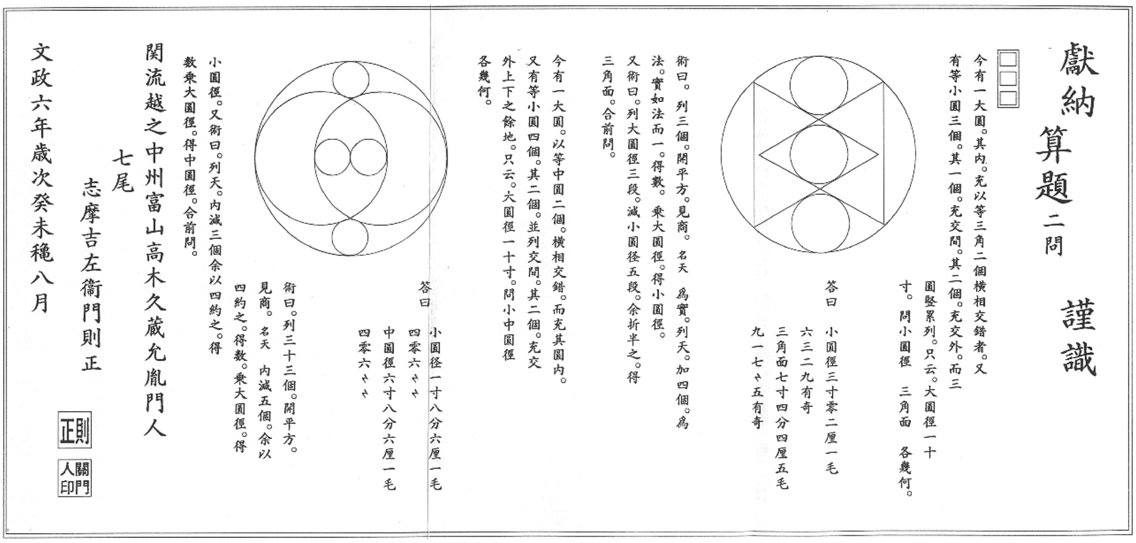


FIGURE 2.40: Modern transcription of the *sangaku* at Kumakabuto Arakashihiko shrine, Ishikawa prefecture. (H. Kotera[34]).

The Kumakabuto Arakashihiko shrine 久麻加夫都阿良加志比古神社 is located in the Nakajima region of the Noto Peninsula in Ishikawa prefecture. It worships the *Arakashihiko* and *Tsunega-arashito* deities. The original tablet was dedicated to the shrine in August 1823.

2.8.12.1 Sources and Transcription

Due to the poor state of the original Kumakabuto Arakashihiko tablet and photographs of it, the transcriptions in this section have been taken from a transcription provided on H. Kotera's website[35].

An image (though of poor quality) of the original tablet can be found in D.8 in Appendix D. A larger image of the transcription provided by Kotera can be found in D.9.

[34]See http://www.wasan.earth.linkclub.com/isikawa/arakasiganbun.html
[35]See http://www.wasan.earth.linkclub.com/isikawa/arakasiganbun.html

2.8.12.2 Accompanying Text

Before Problems

獻納	Dedication
算額二問	Sangaku - two problems
謹識	Humbly written

After Problems

関流越之中州富山高木久蔵允胤門人	Seki School, Takagi Branch, Nakasa Toyama
七尾	Nanao (region)
文六政年歲次未穐八月	Bunsei era 6th year (1823). Year of the Ram. Autumn, August.

2.8.12.3 Kumakabuto Arakashihiko: First Problem

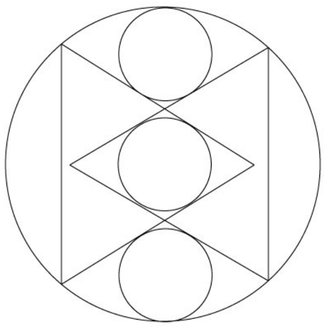


Figure 2.41: First Kumakabuto Arakashihiko problem. (H. Kotera[36]).

Translation

Transcription	Translation
今有一大圓。[37]其内。充以等三角二個横相交錯者。又有等小圓三個。其一個。充交問。其二個。充交外。而三圓竪累列。只云。大圓徑一十寸。問小圓徑　三角面各幾何。	There is one large circle (大). Two equal sized equilateral triangles are inscribed within it which meet each other. Furthermore there are three equal small circles (小). One touches the four sides of the triangles. The other two touch the large circle. The three circles sit on a vertical line. Say the diameter of large circle is 10 *sun*. Problem - what are the diameters of the small circles and the side length of the triangles?

[36]See http://www.wasan.earth.linkclub.com/isikawa/arakasiganbun.html

答日　小圓徑三寸零二厘一毛
六三二九有奇
三角面七寸四分四厘五毛九
一七〻[38]五有奇

Answer: The diameter of the small circles is 3 *sun* 02 *rin* 1 *mo* 6329...
The side length of the triangles is 7 *sun* 4 *bu* 4 *rin* 5 *mo* 91775...

術日。列三個。開平方。見商。
名天爲實[39]。列天。加四個。爲
法[40]。實加法而一[41]。得数。乘
大圓徑。得小圓徑。又術日。
列[42]。大圓徑三段減小圓徑五段。
余折半之。得三角面。合前問。

Technique: Put 3 *ko* and take the square root. Name this *heaven* and make it the dividend. Put *heaven* and add 4 *ko*. Make it the divisor. Divide into dividend. Obtain number. Multiply by the diameter of the large circle. Obtain the diameter of the small circles. Further technique: Put the diameter of the large circles and multiply by 3. Subtract the diameter of the small circles multiplied by 5. Halve the value. Obtain the length of the triangles as previously required.

Translation Notes

The problem text contains the characters *tamejitsu* 爲實, *tamehou* 爲法, and *jitsukahoujiichi* 實加法而一 which appear to derive directly out of Chinese mathematical texts. This particular phrasing can be traced back to the famous Chinese *Nine Chapters on the Mathematical Art* 九章算術. For instance, the following text taken from a section of the *Nine Chapters* relating to calculation of fractions contains all these characters

> 術日; 母互乘子，以少減多，絵爲實。母相乘爲法。實如法而一。[91, p. 281]

The characters *tamejitsu* 爲實 assign a certain value as the dividend, while *tamehou* 爲法 assigns the divisor. *Jitsukahoujiichi* 實加法而一 instructs that division of these values should be carried out.

[38]Round dots '。' are used as a full stop in the modern Japanese language. In the *tenzan jutsu* symbolic manipulation system discussed in section 3.2.1, a large circle ◯ is used to mark the end of a section of text. Therefore instances of this circle have been interpreted as a full stop.

[39]The character 〻 is used in Japanese to signify the repetition of the previous character [95, p. 277]. It is an older variant of the more modern character 々. As it follows the number seven, in this context the tablet text 七〻 can be translated as 77.

[40]The characters *tamejitsu* 爲實 assign a certain value as the dividend, and appear to derive directly from Chinese mathematical texts. See Translation Notes for more.

[41]The characters *tamehou* 爲法 assign a certain value as the divisor.

[42]*Jitsukahoujiichi* 實加法而一 instructs that division of two values should be carried out. In the context of the text, these are the values assigned as the dividend and divisor by the terms *tamejitsu* 爲實 and *tamehou* 爲法.

[43]The character *tan* 段 is uncommon in *sangaku* and translates to 'grade', 'rank', or 'level'. It appears after numerical values, and in this context can be interpreted as indicating a value is multiplied by the given number. The text 大圓徑三 has been translated as a multiplication of the large circles by three.

Technical Analysis

Rendering 1 - Procedure

Problem Say the large circle diameter is 10 *sun*. What are the diameters of the small circles and the side length of the triangles?

Answer Small circles = 3.0216329... *sun*

Triangle sides = 7.44591775... *sun*

Solution $\sqrt{3} \to heaven$

$Heaven \div (heaven + 4)$

$(Heaven \div (heaven + 4)) \times$ large circle = small circle

Small circle = 3.0216329 *sun*

Large circle $\times$ 3

(Large circle $\times$ 3) $-$ (small circle $\times$ 5)

((Large circle $\times$ 3) $-$ (small circle $\times$ 5)) $\div$ 2 = triangle side

Triangle side = 7.44591775 *sun*

Rendering 2 - Formula

Let the diameter of the large circles $= a$, the diameter of the small circles $= x$, and the side length of the triangles $= s$.

Problem Say $a = 10$. What are the values of x and s?

Answer $x = 3.0216329\ldots$

$s = 7.44591775\ldots$

Solution $x = \dfrac{\sqrt{3}}{\sqrt{3}+4} \cdot a$ and $s = \dfrac{3a-5x}{2}$

Small Circle - Modern Analysis

In Figure 2.42, let O be the centre of the large circle and R its radius. Let r be the radius of the small circles. Assume lines OC and CA which represent the distance vertically between the smaller circles, AB and AD which are equal to r, and OD which is equal

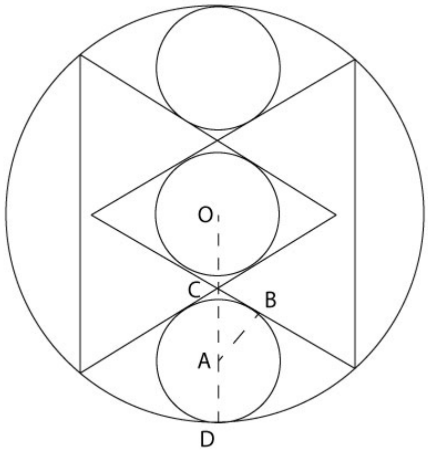


Figure 2.42: Kumakabuto Arakashihiko: First problem analysis

to R. Assume an 30:60:90 triangle $\triangle ABC$. From Figure 2.42,

$$2R = 2r + 4AC \tag{2.50}$$

In $\triangle ABC$

$$AB = r$$

$$r = \frac{\sqrt{3}}{2} \cdot AC$$

$$AC = \frac{2}{\sqrt{3}} \cdot r \tag{2.51}$$

From this and (2.50), it follows

$$2R = 2r + 4(\frac{2}{\sqrt{3}}) \cdot r$$

$$R = (1 + \frac{4}{\sqrt{3}}) \cdot r$$

$$R = \frac{4 + \sqrt{3}}{\sqrt{3}} \cdot r$$

$$2r = \frac{\sqrt{3}}{4 + \sqrt{3}} \cdot 2R = 3.021694793 \tag{2.52}$$

Triangle Side - Modern Analysis

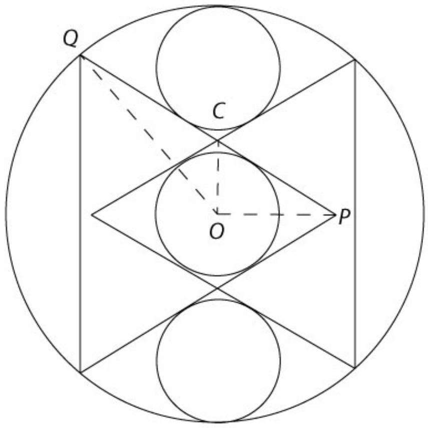


FIGURE 2.43: Kumakabuto Arakashihiko: First problem analysis cont.

As in Figure 2.43, assume a triangle $\triangle OPQ$ which consists of two triangles $\triangle COQ$ and $\triangle COP$. From Figure 2.43

$$OC = \frac{2}{\sqrt{3}} \cdot r$$
$$OPQ = 30^\circ$$
$$OP = \sqrt{3} \cdot OC$$
$$OQ = R$$

The law of cosines can be applied to find the length QP

$$s = QP$$
$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$
$$R^2 = s^2 + (2r)^2 - 2s(2r)\left(\frac{\sqrt{3}}{2}\right)$$
$$R^2 = s^2 + 4r^2 - (2\sqrt{2}r)s \tag{2.53}$$

Subtracting R^2 from both sides

$$s^2 + 4r^2 - (2\sqrt{3}r)s - R^2 = 0$$
$$s^2 - (2\sqrt{3}r)s - (R^2 - 4r^2) = 0 \tag{2.54}$$

This takes the form of a quadratic equation, and the quadratic formula can be applied to solve for s

$$\begin{aligned} s &= \frac{2\sqrt{3}r \pm \sqrt{12r^2 + 4(R^2 - 4r^2)}}{2} && (2.55) \\ &= \frac{2\sqrt{3}r \pm \sqrt{4R^2 - 4r^2}}{2} \\ &= \sqrt{3}r \pm \sqrt{R^2 - r^2} \\ &= \sqrt{3}r + \sqrt{R^2 - r^2} && (2.56) \end{aligned}$$

Substituting previously found value for r from (2.52) to locate $R^2 - r^2$

$$\begin{aligned} R^2 - r^2 &= \left(1 - \frac{3}{(4+\sqrt{3})^2}\right) \cdot R^2 \\ &= \left(1 - \frac{3}{19 + 8\sqrt{3}}\right) \cdot R^2 \\ &= \frac{16 + 8\sqrt{3}}{19 + 8\sqrt{3}} \cdot R^2 \\ &= \frac{4(1+\sqrt{3})^2}{(4+\sqrt{3})^2} \cdot R^2 && (2.57) \end{aligned}$$

This gives

$$s = \sqrt{3}r + \frac{2(1+\sqrt{3})}{4+\sqrt{3}} \cdot R \qquad (2.58)$$

Expressing this fully in terms of r

$$\begin{aligned} s &= \sqrt{3}r + 2\frac{\sqrt{3}}{4+\sqrt{3}} \cdot R + \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{4+\sqrt{3}} \cdot R \\ &= \left(\sqrt{3} + 2 + \frac{2}{\sqrt{3}}\right) r \\ &= \left(2 + \frac{5}{3}\sqrt{3}\right) r = 7.383135547 && (2.59) \end{aligned}$$

Comments

The answer provided on the tablet for the small circle diameter of 3.0216329 *sun* is approximately correct to five decimal places. This result appears to have been obtained through the use of 1.732 as an approximation for $\sqrt{3}$, as substituting this value for $\sqrt{3}$ produces $\frac{1.732}{4+1732} \cdot 10 = 3.021632938$ *sun*.

For the triangle side length, the formula and numerical value obtained in the modern approach differ to that on the tablet. Dr Fukagawa, who has also worked on this problem, has also obtained the same numerical result of 7.383135547 *sun* as found in the modern approach, indicating the author of the tablet likely made a calculation error which led to this different formula and result[44].

2.8.12.4 Kumakabuto Arakashihiko: Second Problem

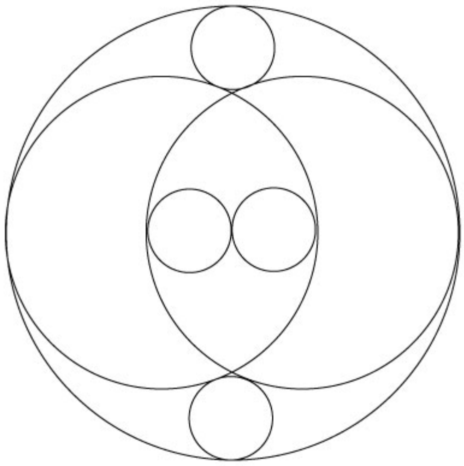


FIGURE 2.44: Second Kumakabuto Arakashihiko problem. (H. Kotera[45]).

Translation

TRANSCRIPTION	TRANSLATION
今有一大圓。[46]以等中圓二個。横相交錯[47]。而充其圓内。又有等小圓四個。其二個。並列交[48]問。其二個。充交外上下之餘地[49]。只云大圓徑一十寸。問小中圓徑各幾何	There is one large circle (大). Two medium circles (中) are internally touching the large circle horizontally. Furthermore there are four small circles (小) of equal size. Two of them are touching one another. Two other circles are touching the outside (外) circle internally respectively at its top and bottom. Say the diameter of the large circle is 10 *sun*. Problem - what are the diameters of the medium and small circles?

[44]From personal communication.
[45]See http://www.wasan.earth.linkclub.com/isikawa/arakasiganbun.html

答日　小圓徑一寸八分六厘一毛四零六〻〻	Answer: The diameter of the small circle is 1 *sun* 8 *bu* 6 *rin* 1 *mo* 40666
中圓徑六寸八分六厘一毛四零六〻〻	The diameter of the medium circle is 6 *sun* 8 *bu* 6 *rin* 1 *mo* 40666
術日　列三十三個。開平方。見商。名天　内減五個。余以四約之。得数。乘大圓徑。得小圓徑。又術日。列天。内減三個余以四約之。得数乘大圓徑。得中圓徑。合前問	Technique: Put 33 *ko* and take the square root. Name this *heaven*. Inside subtract 5 *ko* and divide by means of 4. Obtain value. Multiply by the diameter of the large circle. Obtain the diameter the of small circle. Further technique: Subtract 3 *ko* from *heaven*. Divide by means of 4 and multiply by the diameter of the large circle. Obtain the diameter of the medium circle as previously required.

Translation Notes

As with the first problem, the transcriptions in this section have been taken from a transcription provided on Kotera Hiroshi's website[50].

Technical Analysis

RENDERING 1 - PROCEDURE

Problem Say the large circle diameter is 10 *sun*. What are the diameters of the small circles and the medium circles?

Answer Small circles = 1.86140666 *sun*

Medium circles = 6.86140666 *sun*

[46] See footnotes to problem 1 in section 2.8.12.3 for details regarding the character '。'.

[47] 横相交錯 contains the characters 横 meaning 'horizontal', 相'mutual', and 交錯'entangled'. Together they indicate the middle circles are positioned horizontally.

[48] In the Japanese *Kanbunpoukiso hontou ni wakaru kanbun nyumon*, Kaji Nobuyuki states that 並列 can be read as 与 or 與 [61, p. 350]. Kaji writes that these translate "closest to 'with"' in meaning [61, p.128]. The following character 交 indicates mixing or association. I have interpreted these to mean - given the context of the diagram - that the circles referred to by these characters are touching.

[49] The character 充 indicates 'filling' and 外 refers to the larger circle. 上下之餘地 contains the characters 上下 for 'upper and lower', the possessive particle 之, and the characters for remainder/excess 餘 and ground 地。I translated this section as meaning that the last two circles combined fill the excess space of the upper and lower regions of the larger circle (that is, the area between the middle two circles and the large circle).

[50] See http://www.wasan.earth.linkclub.com/isikawa/arakasiganbun.html

Solution $\sqrt{33} \to heaven$

$(Heaven - 5) \div 4$

$((Heaven - 5) \div 4) \times$ large circle = small circle

Small circle = 1.86140666 *sun*

$Heaven - 3$

$(Heaven - 3) \div 4$

$((Heaven - 3) \div 4) \times$ large circle = medium circle

Medium circle = 6.86140666 *sun*

Rendering 2 - Formula

Let the diameter of the large circles = a, the diameter of the small circles = x, and the diameter of the medium circles = y.

Problem Say $a = 10$. What are the values of x and y?

Answer $x = 1.8614066$

$y = 6.86140666$

Solution $x = \dfrac{\sqrt{33} - 5}{4} \cdot a$ and $y = \dfrac{\sqrt{33} - 3}{4} \cdot a$

Small Circle - Modern Analysis

As in Figure 2.45, let the radius of the large circle be R, the radius of the medium circles be r_2, and the radius of the small circles be r_3. Assume a triangle $\triangle ABC$. In Figure 2.45

$$AC = z$$
$$BC = r_2 + r_3$$
$$AB = R - r_2$$
$$2R = 2(2r_2) - 2(2r_3) \tag{2.60}$$

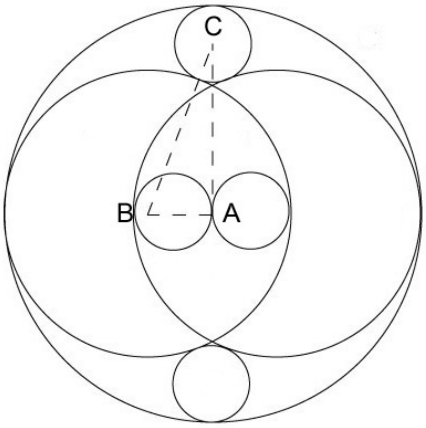


FIGURE 2.45: Kumakabuto Arakashihiko: Second problem analysis

Applying the Pythagorean theorem to $\triangle ABC$ gives

$$(r_2 + r_3)^2 = (R - r_2)^2 + z^2 \tag{2.61}$$

From (2.60)

$$\begin{aligned} 2R &= 2(2r_2) - 2(2r_3) \\ R + 2r_3 &= 2r_2 \end{aligned} \tag{2.62}$$

Combining (2.61) and (2.62)

$$\begin{aligned} R &= z + r_3 \\ (r_2 + r_3)^2 &= (R - r_2)^2 + (R - r_3)^2 \\ {r_2}^2 + 2r_2r_3 + {r_3}^2 &= (R^2 - 2Rr_2 + {r_2}^2) + (R^2 - 2Rr_3 + {r_3}^2) \\ 2R^2 - 2(r_2 + r_3)R - 2r_2r_3 &= 0 \\ R^2 - (r_2 + r_3)R - r_2r_3 &= 0 \end{aligned} \tag{2.63}$$

Combining (2.62) and (2.63)

$$\begin{aligned} 2(r_2 - r_3)^2 - 2(r_2 - r_3)(r_2 + r_3) - r_2r_3 &= 0 \\ (4{r_2}^2 - 8r_2r_3 + 4{r_3}^2) - 2({r_2}^2 - {r_3}^2) - r_2r_3 &= 0 \\ 2{r_2}^2 - 9r_2r_3 + 6{r_3}^2 &= 0 \end{aligned} \tag{2.64}$$

$$\frac{r_2}{r_3} = \frac{9 \pm \sqrt{81 - 48}}{4}$$
$$\frac{r_2}{r_3} = \frac{9 \pm \sqrt{33}}{4}$$
$$r_2 = \frac{9 + \sqrt{33}}{4} \cdot r_3 \tag{2.65}$$

From (2.60) and (2.64)

$$R = 2(r_2 - r_3)$$
$$R = 2\left(\frac{9 + \sqrt{33}}{4} - 1\right)$$
$$R = \frac{5 + \sqrt{33}}{2} \cdot r_3$$

$$r_3 = \frac{2}{5 + \sqrt{33}} \cdot R \tag{2.66}$$
$$r_3 = \frac{2(\sqrt{33} - 5)}{33 - 25} \cdot R$$
$$r_3 = \frac{\sqrt{33} - 5}{4} \cdot R \qquad 2r_3 = \frac{\sqrt{33} - 5}{4} \cdot 2R = 1.861406616 \tag{2.67}$$

Medium Circle - Modern Analysis

For r_2

$$r_2 = \frac{9 + \sqrt{33}}{4} \cdot \left(\frac{\sqrt{33} - 5}{4}\right) \cdot R$$
$$r_2 = \frac{4\sqrt{33} + (33 - 45)}{16} \cdot R$$
$$r_2 = \frac{4\sqrt{33} - 12}{16} \cdot R$$
$$r_2 = \frac{\sqrt{33} - 3}{4} \cdot R \qquad 2r_2 = \frac{\sqrt{33} - 3}{4} \cdot 2R = 6.861406616 \tag{2.68}$$

Comments

The answers provided on the tablet for the small and medium circles are approximately correct. The value for the small circle given by the author is 1.86140666 *sun* while the value found in the modern analysis is 1.861406616 *sun*. For the medium circle, there is again a small difference of one value. If the author was using an approximation of 5.74456264 for $\sqrt{33}$, this would produce 1.8614066 *sun* and 6.8614066 *sun*. The author may have used this approximation and added an extra 6 by mistake. However it is also possible that the author used 5.744562647 as their approximation and made a small mistake in their calculation, placing 6 instead of 1 towards the end of the values. As well as this, one must consider whether this may have been an error made by the transcriber of the tablet.

2.8.13 Suwa Sangaku

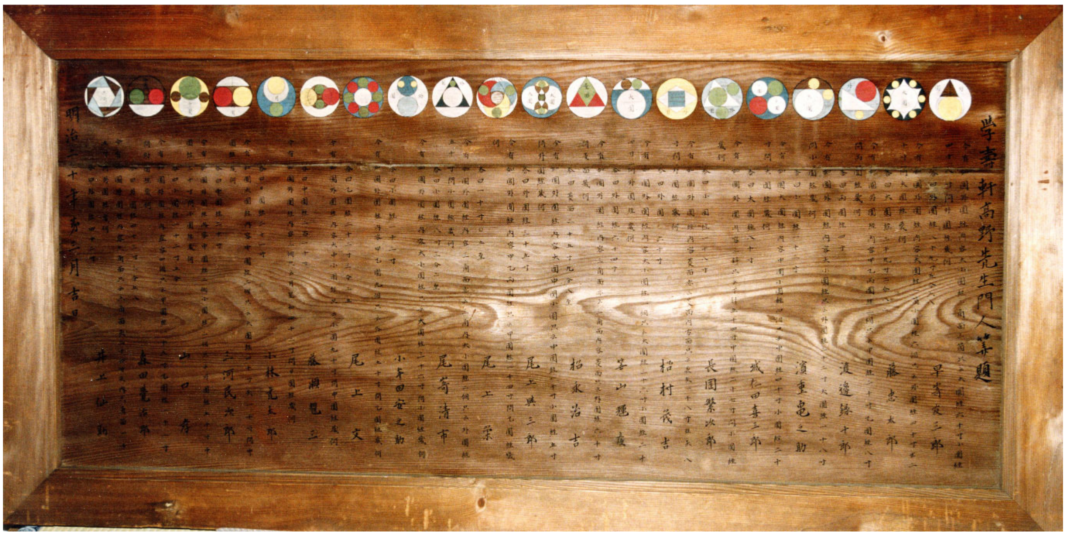


FIGURE 2.46: The *sangaku* at Suwa shrine, Nagasaki prefecture. (H. Kotera[51]).

The Suwa shrine 諏訪神社 is located in Nagasaki prefecture. It contains a *sangaku* dedicated in 1887. The tablet contains twenty different problems constructed by nineteen students of the Takasaki school. After each problem, the name of the student who constructed it is listed. However, due to the difficulty with transcribing these names, they have been obmitted from the translations. One feature of this *sangaku* is that none of its problems contain a technique section with a formula. Here focus appears placed on finding numerical answers, with the observer is left to find their own formula rather than understanding that of the author. Because there is no formula section, no second rendering in the technical analysis is given.

2.8.13.1 Sources and Transcription

Thought this tablet is beautiful, unfortunately the photographs available are of a poor quality, making transcription difficult. Being unable to photograph this tablet myself, for my transcriptions I have relied on photographs from Kotera Hiroshi's *wasan* website and the article 'Review of the Suwa Sangaku' 長崎県の和算の概説 by Hinoto Yonemitsu [104] which contains transcriptions for each problem on the tablet along with a modern Japanese translation. While transcriptions are provided by Yonemitsu, he does not solve the problems.

I have supplemented characters from Yonemitsu's transcriptions as necessary when identification was difficult due to the poor quality of the tablet images. By combining

[51]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

these two sources I have produced the transcriptions within this section. A larger version of the image from H. Kotera's site can be found in D.10 of Appendix D.

2.8.13.2 Accompanying Text

Before problems

学壽高崎先生門人算額	Congratulations for learning Takasaki teacher school sangaku

Text after problems:

明治二十年 ☐ 三月吉日	Meiji era 20th year. March. On a happy day.

2.8.13.3 Suwa: First Problem

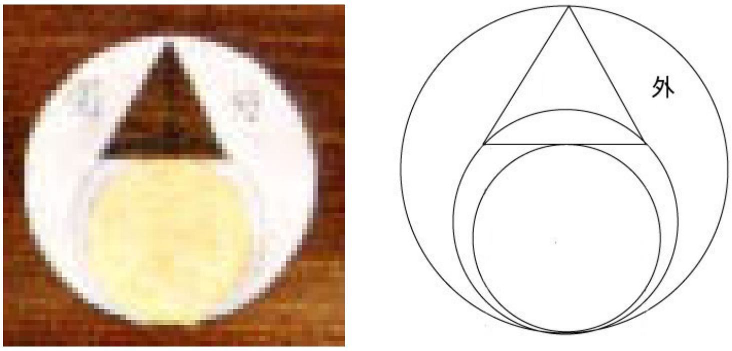


Figure 2.47: Left: First Suwa problem. Right: Transcription. (H. Kotera[52]).

Translation

Transcription	Translation
今有如圖外圓径内容大小圓径三角面三個只云大圓径六十寸小圓径四十八寸問外圓径幾何	As in the diagram, there is an outer circle (外) which contains a large circle (大), a small circle (小), and an equilateral triangle. Say the diameter of the large circle is 60 *sun* and the diameter of the small circle is 48 *sun*. Problem - what is the diameter of the outer circle?
答日　外圓径八十九寸五分六八	Answer: The diameter of the outer circle is 89 *sun* 5 *bu* 68

[52]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

Technical Analysis

Rendering 1 - Procedure

Problem Say inside the outer circle is a large circle of diameter 60 *sun* and a small circle of 48 *sun*. What is the diameter of the outer circle?

Answer Outer circle diameter = 89 *sun* 5 *bu* 68

Solution

Modern Analysis

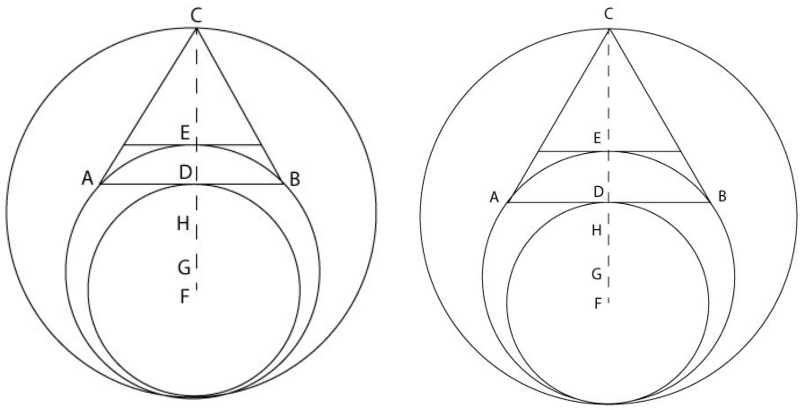


Figure 2.48: Suwa: First problem analysis. Left: Figure proportions as per tablet diagram. Right: Figure proportions as per given values.

Figure 2.48 gives a transcription of the tablet diagram on the left, and a version of the diagram created by using all the exact lengths in the *sangaku* in Adobe Illustrator on the right.

In Figure 2.48, let the centre of the given outer circle be H, the centre of the large circle be G and the centre of the small circle be F. Say the diameter of the outer circle is d_1, the diameter of the large circle is d_2, and the diameter of the small circle is d_3. Assume the triangle drawn on the tablet as $\triangle ABC$. When examining the diagram, regardless of which diagram in Figure 2.48 is observed, since triangle $\triangle ABC$ is equilateral d_1 can be found through combining the triangle altitude CD with d_3.

To find the altitude CD, the triangle side length AB is first calculated. Since the side AB also forms a chord in d_2, its value can be found by applying a formula to find half the length of a chord when the sagitta and radius of a circle are known. Applying this, where s is the sagitta and l is half the length of the chord

$$
\begin{aligned}
&l = AB \\
&s = d_2 - d_3 \\
&l = \sqrt{d_2 s - s^2} \\
&l = 24 \\
&AB = 48
\end{aligned}
\tag{2.69}
$$

From (2.69) CD can be calculated and added with d_3 to find d_1 as follows

$$
\begin{aligned}
&CD = AB \times \frac{\sqrt{3}}{2} \\
&d_1 = CD + d_3 \\
&d_1 = 89.569
\end{aligned}
\tag{2.70}
$$

Comments

While the *sangaku* diagram is not metrically precise, it still manages to illustrate the details required to make the above method work. As long as the triangle altitude CD and the diameter d_3 of the small circle c together form the diameter of the outer circle, and the sagitta is $(b - c)$, the circles b and c can be drawn at any size, as the method uses only geometrical relationships and does not rely on particular lengths.

With the differing diagrams, it is possible the author originally drew the diagram as it appears on the tablet, being were aware that as long as they represented the relationships accurately the sizes of the circles could vary. However the author may have also created the diagram as seen in Illustrator first and then altered the circle sizes to induce contemplation of the geometrical principles behind the solution and problem.

2.8.13.4 Suwa: Second Problem

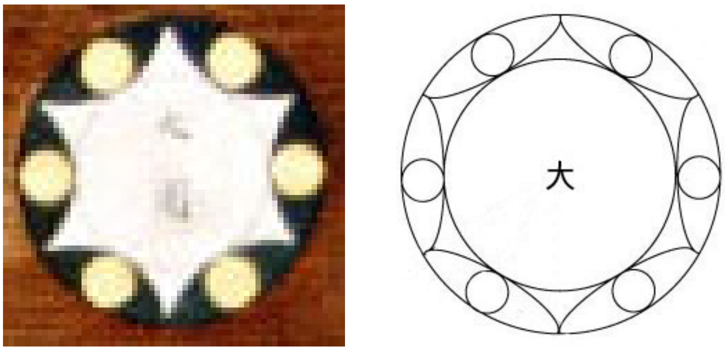


FIGURE 2.49: Left: Second Suwa problem. Right: Transcription. (H. Kotera[53]).

Translation

TRANSCRIPTION	TRANSLATION
今有如圖外圓径内容大圓径一個小圓径六個只云外圓径四十寸弦二十寸問大圓径幾何	As in the diagram, there is an outer circle (外) which contains one large circle (大) and six small circles (小). Say the diameter of the outer circle is 40 *sun* and the chord [between two points of the star] is 20 *sun*. Problem - what is the diameter of the large circle?
答日　大圓径二十九寸二分八二	Answer: The diameter of the large circle is 29 *sun* 2 *bu* 82

Technical Analysis

RENDERING 1 - PROCEDURE

Problem Say the diameter of the outer circle is 40 *sun* and the chord is 20 *sun*. What is the diameter of the large circle?

Answer Large circle diameter = 29 *sun* 2 *bu* 82

Solution

[53]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

Modern Analysis

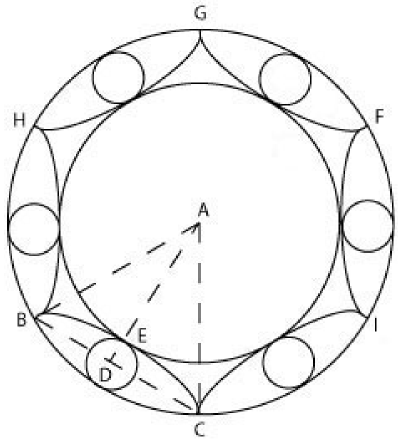


FIGURE 2.50: Suwa: Second problem analysis

In Figure 2.50, let the centre of the given outer circle be A with a radius R, the centre of the small circles be D with radius r_1, and the centre of the large circle be A with radius r_2. Although the chord is not drawn, due to the fact it is of length 20 and the diameter of the outer circle is 40, it likely forms one side of an hexagon inscribed in the outer circle, for the sides of an inscribed hexagon equal the radius of the circle they inscribe. This would place the chord between two of the six points of the star BC, CE, EF, FG, GH, or HB. By treating one of these as the location of the chord, an answer which matches that on the tablet can be produced.

To note, the tablet text itself provides little information about the figures in the diagram, and the information given is repetitive. For instance, when a chord equals one side of an inscribed hexagon, its the numerical value is not required, for the geometry of the hexagon determines its sides are equal to the radius of the circle. However, since the chord is not drawn on the diagram, the numerical value may have been supplied to enable the observer to determine the location of the chord.

From Figure 2.50, where BC is assumed as the chord, since the chord is 20 and R is 20, a 60:60:60 triangle $\triangle ABC$ with an altitude AD can be formed.

Firstly, the value of r_1 can be obtained by calculating the length of the altitude AD and rearranging

$$AB = AC = BC = R = 20$$

$$AE = r_1$$

$$
\begin{aligned}
DE &= r_2 \\
AD &= \frac{\sqrt{3}}{2} \times 20 = 20 - r_1 \\
r_1 &= 20 - \left(\frac{\sqrt{3}}{2} \times 20 \right)
\end{aligned}
\tag{2.71}
$$

Having the value for r_1, now r_2 can be found by subtracting the value of r_1 from the length AD

$$
\begin{aligned}
r_2 &= R - 2r_1 \\
&= 20 - 2\left(20 - \left(\frac{\sqrt{3}}{2} \times 20 \right) \right) \\
&= 14.64101615 \\
2r_2 &= 29.2820323
\end{aligned}
\tag{2.72}
$$

Comments

The omitting of the sagitta from the diagram creates a disconnection between the diagram and text, as the diagram does not accurately represent what is described in the text. This alludes to the need for greater contemplation of the problem by the observer. The observer must first infer the positioning of the chord, causing them to consider the diagram and text more deeply than they might if the chord was already drawn. The inclusion of a drawn chord in one of the positions of the sides of a hexagon $BCEFGH$ would also create a lack of visual symmetry in the problem. This could be overcome however by adding in every chord between the points of the star. However, by adding in such lines a hexagon with the chords as its side length would be formed, which may obstruct the intended visual display of the problem by the author.

2.8.13.5 Suwa: Third Problem

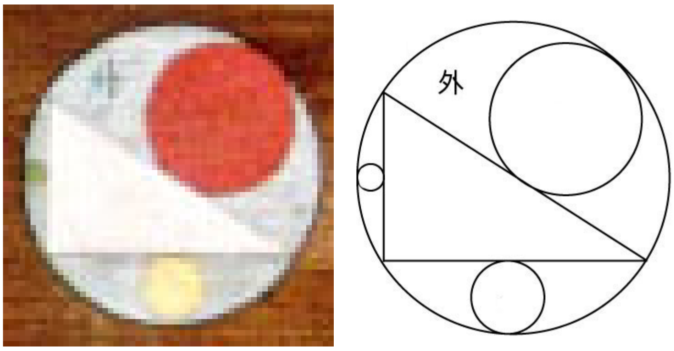


FIGURE 2.51: Left: Third Suwa problem. Right: Transcription. (H. Kotera[54]).

Translation

TRANSCRIPTION	TRANSLATION
今有如圖外圓径内容甲乙丙圓径三個只云甲圓径二十寸乙圓径八寸問丙圓径幾何	As in the diagram, there is an outer circle (外) which contains three circles *kō* 甲, *otsu* 乙, and *hei* 丙. Say the diameter of circle *kō* 甲 is 20 *sun* and the diameter of circle *otsu* 乙 is 8 *sun*. Problem - what is the diameter of circle *hei* 丙?
答曰　丙圓径四寸	Answer: The diameter of circle *hei* 丙 is 4 *sun*

Technical Analysis

RENDERING 1 - PROCEDURE

Problem Three circles are inside an outer circle. Say the diameter of the circle *kō* 甲 is 20 *sun* and the diameter of circle *otsu* 乙 is 8 *sun*. What is the diameter of circle *hei* 丙?

Answer Circle *hei* 丙 = 4 *sun*

Solution

Modern Analysis

In Figure 2.52, assume circle *kō* 甲 is the largest internal circle a, *otsu* 乙 is b, and *hei* 丙 is c, with respective diameters d_1, d_2, and d_3. Let the radius of the outer circle be R. Assume there is a right angle triangle $\triangle ABC$ containing a rectangle $CEDF$.

[54]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

From the right of Figure 2.52, the diameter of the large circle labelled a visually appears equal to the radius of the outer circle. By assuming this, the answer on the tablet can be obtained.

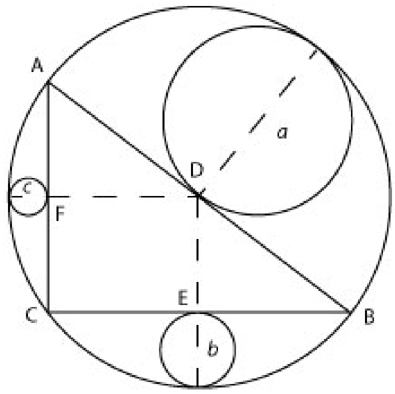


FIGURE 2.52: Suwa: Third problem analysis

From Figure 2.52,

$$R = d_1$$
$$DE = R - d_2$$
$$DE = \frac{AC}{2}$$

Knowing R and AC, d_3 can be found through a sagitta calculation where l is half the length of the chord and s is the sagitta

$$s = R - \sqrt{R^2 - l^2}$$
$$s = R - \sqrt{R^2 - DE^2}$$
$$s = 20 - \sqrt{20^2 - 12^2}$$
$$s = d_3 = 4 \tag{2.73}$$

2.8.13.6 Suwa: Fourth Problem

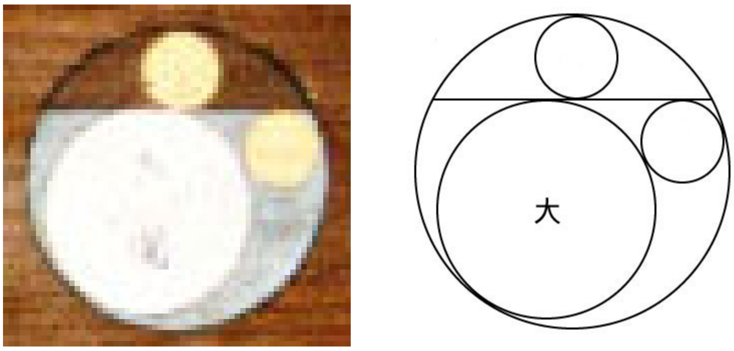


FIGURE 2.53: Left: Fourth Suwa problem. Right: Transcription. (H. Kotera[55]).

Translation

TRANSCRIPTION	TRANSLATION
今有如圖外圓径内容大小圓径只云外圓径三十二寸大圓径一十八寸問小圓径幾何	As in the diagram, there is an outer circle (外) which contains a large circle (大) and small circles (小). Say the diameter of the outer circle is 32 *sun* and the diameter of the large circle is 18 *sun*. Problem - what is the diameter of the small circles?
答日　小圓径一十二寸	Answer: The diameter of the small circles is 12 *sun*

Technical Analysis

RENDERING 1 - PROCEDURE

Problem Inside the outer circle of diameter 32 *sun* is a large circle with a diameter of 18 *sun* and small circles. What is the diameter of the small circles?

Answer Small circle diameter = 12 *sun*

Solution

[55]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

Modern Analysis

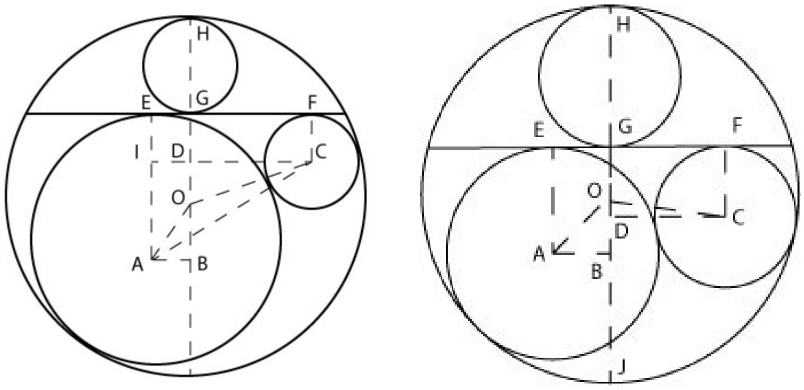


Figure 2.54: Suwa: Fourth problem analysis. Left: Figure proportions as per tablet diagram. Right: Figure proportions as per given values.

In Figure 2.54, the left-hand shows the transcribed diagram with the proportions as given on the tablet, and the right-hand those produced from the values from the text in Illustrator. The small circles in the left diagram are nearly half the size of those in the right, and the large circle is drawn smaller. In Figure 2.53, one can see the outer circle and small circles are also not labelled on the diagram, but given the sizes of the circles and use of the size descriptive labels 'large' and 'small', it is assumed the outer circle corellates to a in the right of Figure 2.54 and the small circles to c.

As in the right of Figure 2.54, say the centre of the given outer circle is O. Assume a diameter HJ consisting of five line segments HG, GO, OD, DB, and BJ. Also assume two right angle triangles $\triangle OAB$ and $\triangle OCD$, and a perpendicular line EF. Say the diameter of the outer circle is d, the diameter of the large circle is d_1, and the diameter of the small circles is d_2.

From triangle $\triangle OAB$

$$
\begin{aligned}
OA &= \frac{d}{2} - \frac{d_1}{2} \\
OB &= \frac{d_1}{2} + d_2 - \frac{d}{2} \\
AB^2 &= OA^2 - OB^2 \\
&= \left(\frac{d}{2} - \frac{d_1}{2}\right)^2 - \left(\frac{d_1}{2} + d_2 - \frac{d}{2}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(d-d_1)^2}{4} - \frac{(d_1+2d_2-d)^2}{4} \\
&= \frac{d^2-2dd_1+{d_1}^2-{d_1}^2+2dd_1-d^2-4d_1d_2+4dd_2-4{d_2}^2}{4} \\
&= \frac{4dd_2-4d_1d_2-4{d_2}^2}{4} \\
&= d_2(d-d_1-d_2) \qquad (2.74)
\end{aligned}
$$

From triangle $\triangle OCD$

$$
\begin{aligned}
OC &= \frac{d}{2} - \frac{d_2}{2} \\
OD &= \frac{d_2}{2} + b - \frac{d}{2} \\
CD^2 &= OC^2 - OD^2 \\
&= \left(\frac{d}{2} - \frac{d_2}{2}\right)^2 + \left(\frac{d_2}{2} + b - \frac{d}{2}\right)^2 \\
&= \frac{(d-d_2)^2}{4} - \frac{(d_2+2d_2-d)^2}{4} \\
&= \frac{{d_2}^2-2dd_2+d^2-9{d_2}^2+6dd_2-d^2}{4} \\
&= \frac{4dd_2-8{d_2}^2}{4} \\
&= d_2(d-2d_2) \qquad (2.75)
\end{aligned}
$$

From triangle $\triangle ACI$

$$
\begin{aligned}
CI^2 &= AC^2 - AI^2 \\
&= \left(\frac{d_1+d_2}{2}\right) - \left(\frac{d_1-d_2}{2}\right) \\
&= (\sqrt{d_1d_2})^2 \\
&= d_1d_2 \qquad (2.76)
\end{aligned}
$$

In Figure 2.54, AB and CD together form the line CI. Due to this they can be substituted into (2.76) as follows

$$\begin{aligned}
&(AB + CD)^2 = CI^2 = d_1 d_2\\
&AB^2 + 2AB \cdot CD + CD^2 = d_1 d_2\\
&d_2(d - d_1 - d_2) + d_2(d - 2d_2) + 2AB \cdot CD = d_1 d_2\\
&2AB \cdot CD = d_1 d_2 - d_2(d - d_1 - d_2) - d_2(d - 2d_2) \qquad (2.77)
\end{aligned}$$

Having established the values for AB^2 and CD^2 in (2.74) and (2.75), these values can be substituted for $2AB \cdot CD$ by squaring both sides of (2.77)

$$\begin{aligned}
&(2AB \cdot CD)^2 = (d_1 d_2 - d_2(d - d_1 - d_2) - d_2(d - 2d_2))^2\\
&4AB^2 \cdot CD^2 = (d_1 d_2 - d_2(d - d_1 - d_2) - d_2(d - 2d_2))^2\\
&4(d_2(d - d_1 - d_2))(d_2(d - d_2)) = (d_1 d_2 - d_2(d - d_1 - d_2) - d_2(d - 2d_2))^2\\
&4{d_2}^2(d - d_1 - d_2)(d - 2d_2) = (d_1 d_2 - d_2(d - d_1 - d_2) - d_2(d - 2d_2))^2\\
&8{d_2}^4 + {d_2}^3(8d_1 - 12d) + {d_2}^2(4d^2 - 4dd_1) = 9{d_2}^4 + {d_2}^3(12d_1 - 12d) + {d_2}^2(4d_1 - 8dd_1 + 4d^2)\\
&-{d_2}^4 - 4d_1{d_2}^3 + {d_2}^2(4dd_1 - 4{d_1}^2) = 0\\
&{d_2}^2({d_2}^2 + 4d_1 d_2 + 4{d_1}^2 - 4dd_1) = 0\\
&{d_2}^2 \quad \text{or} \quad {d_2}^2 + 4d_1 d_2 + 4{d_1}^2 - 4dd_1 = 0 \qquad (2.78)
\end{aligned}$$

Rearranging to solve for d_2

$$d_2 = 0 \quad \text{or} \quad -2(d_1 + \sqrt{dd_1}) \quad \text{or} \quad 2(\sqrt{dd_1} - d_1) \qquad (2.79)$$

Since d_2 is not negative or equal to zero, the third equation of (2.79) is assumed. This gives the value on the tablet as follows

$$\begin{aligned}
d_2 &= 2(\sqrt{dd_1} - d_1)\\
&= 2(\sqrt{32 \cdot 18} - 18)\\
&= 2(6) = 12 \qquad (2.80)
\end{aligned}$$

To solve the problem using the diagram provided by the author, the same concept is used but the right angle triangle $\triangle OCD$ must be in a slightly different position. As shown in the left diagram of Figure 2.54, the expression of CD^2 becomes $(\frac{d}{2} - \frac{d_2}{2})^2 - (\frac{d}{2} - d_2 - \frac{d_2}{2})^2$. However this still results in the same expression of CD^2 as $d_2(d - 2d_2)$ meaning the same answer can be found.

Comments

While the same concept can be applied to solve the problem given each diagram in Figure 2.54, a slight alteration in the positioning of triangle OCD is required. The providing of a diagram with different sized circles to what should be produced in Illustrator suggests a contemplation of general methods which can be applied to similar diagrams. However, given the change in OCD, there is no absolute generality between the two.

2.8.13.7 Suwa: Fifth Problem

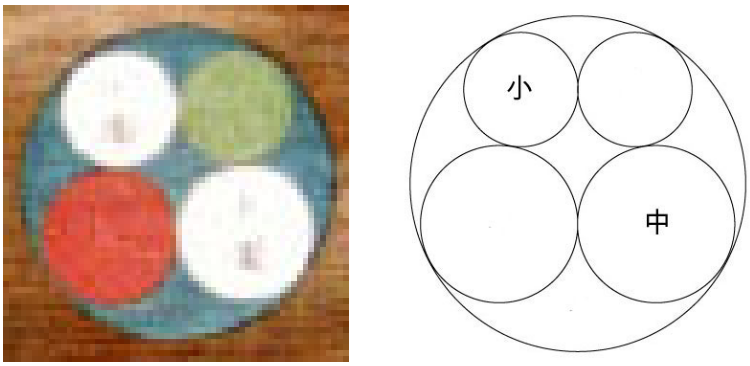


FIGURE 2.55: Left: Fifth Suwa problem. Right: Transcription. (H. Kotera[56]).

Translation

TRANSCRIPTION	TRANSLATION
今有如圖大圓径内中圓小圓径四個只云中圓径四十寸小圓径二十寸問大圓径幾何	As in the diagram, there is an large circle (大) which contains four small circles (小) and medium circles (中). Say the diameter of the medium circles is 40 *sun*, and the diameter of the small circles is 20 *sun*. Problem - what is the diameter of the large circle?
答日大圓径八十寸	Answer: The diameter of the large circle is 80 *sun*

Modern Analysis

In Figure 2.56, a transcription of the original diagram from the tablet is given on the left and the diagram using values from the text created in Illustrator on the right. The circles are labelled such that the medium circles lie below the small. The right diagram gives the diameter of the medium circles as the radius of the large circle, while

[56]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

the left diagram shows these circles as significantly smaller. The diagrams also differ in their presentation of the smaller circles, with the right diagram this time showing them as smaller than the left.

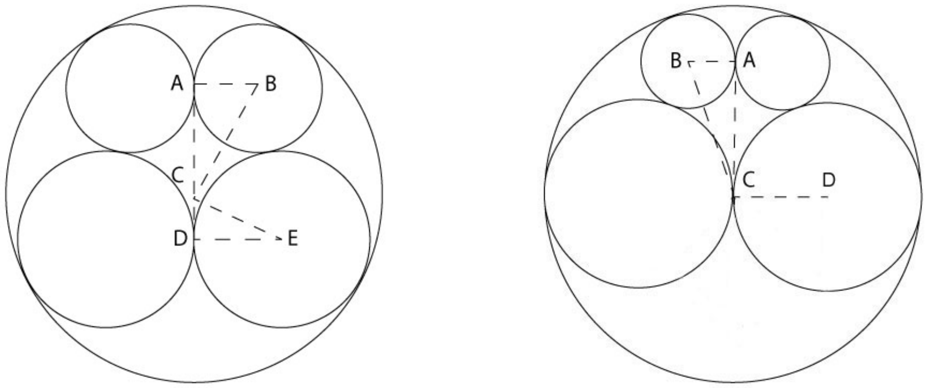


FIGURE 2.56: Suwa: Fifth problem analysis. Left: Figure proportions as per tablet diagram. Right: Figure proportions as per given values.

In the left of Figure 2.56, assume triangles $\triangle ABC$ and $\triangle CDE$, and in the right assume a triangle $\triangle ABC$ and line CD. Let d_1 represent the diameter of the outer circle, d_2 the diameters of the medium circles, and d_3 the diameter of the small circles.

From $\triangle ABC$ and $\triangle CDE$ in the left-hand diagram of Figure 2.56

$$\begin{aligned}
AB &= \frac{d_3}{2} \\
BC &= \frac{d_1}{2} - \frac{d_3}{2} \\
AC &= \frac{1}{2}\sqrt{{d_1}^2 - 2d_1 d_3} \\
CE &= \frac{d_1}{2} - \frac{d_2}{2} \\
DE &= \frac{d_2}{2} \\
CD &= \frac{1}{2}\sqrt{{d_1}^2 - 2d_1 d_2}
\end{aligned} \tag{2.81}$$

From this it can be determined

$$\begin{aligned}
&AD = AC + CD \\
&AD = \sqrt{d_2 d_3} \\
&\sqrt{d_2 d_3} = AC + CD \\
&\sqrt{d_2 d_3} = \frac{1}{2}\sqrt{d_1{}^2 - 2d_1 d_3} + \frac{1}{2}\sqrt{d_1{}^2 - 2d_1 d_2}
\end{aligned} \tag{2.82}$$

Squaring both sides of (2.82) and then multiplying by 4

$$\begin{aligned}
&d_2 d_3 = \frac{d_1{}^2}{2} - \frac{d_1 d_2}{2} - \frac{d_1 d_3}{2} + \frac{1}{2}\sqrt{(d_1{}^2 - 2d_1 d_3)(d_1{}^2 - 2d_1 d_2)} \\
&4d_2 d_3 = 2d_1{}^2 - 2d_1(d_2 + d_3) + 2d_1\sqrt{(d_1 - 2d_3)(d_1 - 2d_2)}
\end{aligned} \tag{2.83}$$

Dividing by 2 and rearranging

$$2d_2 d_3 + d_1(d_2 + d_3) - d_1{}^2 = d_1\sqrt{(d_1 - 2d_2)(d_1 - 2d_3)} \tag{2.84}$$

Squaring both sides

$$\begin{aligned}
&d_1{}^2 - 2(d_2 + d_3)d_1{}^3 + ((d_2 + d_3)^2 - 4d_2 d_3)d_1{}^2 + 4d_2 d_3(d_2 + d_3) + 4d_2{}^2 d_3{}^2 \\
&= d_1{}^4 - 2(d_2 + d_3)d_1{}^3 + (4d_2 d_3)d_1{}^2
\end{aligned} \tag{2.85}$$

Rearranging gives the following quadratic

$$d_1{}^2[(d_2 + d_3)^2 - 8d_2 d_3] + d_1[4d_2 d_3(d_2 + d_3)] + 4d_2{}^2 d_3{}^2 = 0 \tag{2.86}$$

Since $d_2 = 2d_3$, the coefficients in the quadratic become

$$\begin{aligned}
&(d_2 + d_3)^2 - 8d_2 d_3 = 3d_3{}^2 - 16d_3{}^2 = -7d_3{}^2 \\
&4d_2 d_3(d_2 + d_3) = 4(2d_3{}^2)(d_3) = 24d_3{}^3 \\
&4d_2{}^2 d_3{}^2 = 16d_3{}^2
\end{aligned} \tag{2.87}$$

Then cancelling $d_3{}^2$ throughout and substituting (2.87) into (2.86)

$$\begin{aligned}
&7d_1{}^2 - (24d_3)d_1 - 16d_3{}^2 = 0 \\
&(7d_1 + 4d_3)(d_1 - 4d_3) = 0
\end{aligned} \tag{2.88}$$

This gives the following two solutions

$$
\begin{aligned}
d_1 &= -\frac{4d_3}{7} \quad \text{or} \\
d_1 &= 4d_3
\end{aligned}
\tag{2.89}
$$

Using the second solution, the answer found on the tablet can be produced

$$
\begin{aligned}
d_1 &= 4d_3 \\
d_1 &= 4(20) \\
d_1 &= 80
\end{aligned}
\tag{2.90}
$$

Comments

The two different diagrams associated with this problem require slightly different approaches. From the author's diagram, a general method involving two right angle triangles and the Pythagorean theorem can be found. While this can be applied to the right-hand diagram, since this diagram is not able to produce the second triangle $\triangle CDE$ the same general method can not be obtained from observing it alone. Due to this, it may be the case that the diagram on the tablet or something similar was produced first and used to determine the method, with the values of the circles determined later.

2.8.13.8 Suwa: Sixth Problem

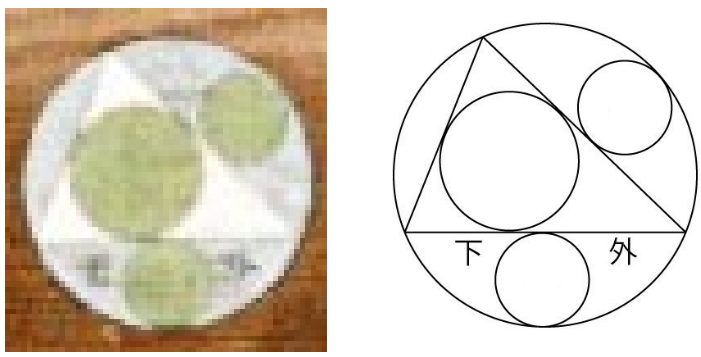


Figure 2.57: Left: Sixth Suwa problem. Right: Transcription. (H. Kotera[57]).

[57]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

Translation

Transcription	Translation
今有如圖外圓径内容三斜只云下斜五十四寸中圓径二十七寸問小圓径幾何	As in the diagram, there is an outer circle (外) which contains a triangle. Say the lower line is 54 *sun* and the diameter of the medium circle is 27 *sun*. Problem - what is the diameter of the small circles?
答日　小圓径一十八寸	Answer: The diameter of the small circles is 18 *sun*

Modern Analysis

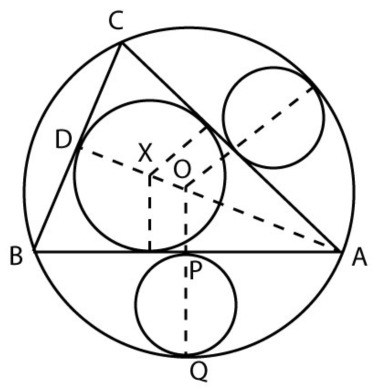


Figure 2.58: Suwa: Sixth problem analysis

In Figure 2.58, let the centre of the given outer circle be O and the centre of the medium circle be X. Say R is the radius of the outer circle, r_1 is the radius of the medium circle, and d_2 is the diameter of the small circles. Let the given isosceles triangle be $\triangle ABC$, and assume it has an altitude AD and base BC.

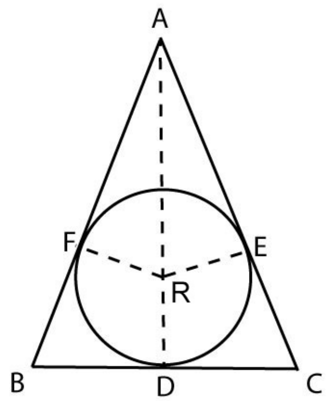


Figure 2.59: Suwa: Sixth Problem Analysis Cont.

To find the value of d_2 from the information on the tablet, assume in Figure 2.59 additional triangles $\triangle AFR$ and $\triangle AER$ inside $\triangle ABC$. Let l represent the given side AB of the triangle $\triangle ABC$ in 2.58. From this,

$$\begin{aligned}
&AB = AC = l \\
&AD = h \\
&RD = RE = RF = \frac{l}{4} \\
&BD = CD = FB = CE = x
\end{aligned} \tag{2.91}$$

The Pythagorean theorem can be applied to $\triangle ABD$

$$x^2 + h^2 = l^2 \quad \text{or} \quad \left(\frac{x}{l}\right)^2 + \left(\frac{h}{l}\right)^2 = 1 \tag{2.92}$$

It can also be applied to triangle $\triangle AFR$ in a similar manner

$$(l-x)^2 + \left(\frac{l}{4}\right)^2 = \left(h - \frac{l}{4}\right)^2 \quad \text{or} \quad \left(1 - \frac{x}{l}\right)^2 + \left(\frac{1}{4}\right)^2 = \left(\frac{h}{l} - \frac{1}{4}\right)^2 \tag{2.93}$$

Focusing on the proportions in the second half of (2.92) and (2.93)

$$X = \frac{x}{l}$$

$$H = \frac{h}{l}$$

$$X^2 + H^2 = 1 \qquad \text{(From 2.92)} \tag{2.94}$$

$$(1-X)^2 + \left(\frac{1}{4}\right)^2 = \left(H - \frac{1}{4}\right)^2 \qquad \text{(From 2.93)} \tag{2.95}$$

Expanding and then squaring (2.95)

$$
\begin{aligned}
1 - 2X + X^2 + \frac{1}{16} &= H^2 - \frac{1}{2}H + \frac{1}{16} \\
1 - 2X + X^2 &= H^2 - \frac{1}{2}H \\
&= (1 - X^2) - \frac{1}{2}H \qquad \text{(from 2.95)} \\
2X - 2X^2 &= \frac{1}{2}H \quad \text{or} \quad 4X(1-X) = H \\
16X^2(1-X)^2 &= H^2 = 1 - X^2 = (1-X)(1+X)
\end{aligned}
\tag{2.96}
$$

Cancelling the factor of $(1 - X)$

$$
\begin{aligned}
&16X^2(1-X) = 1 + X \\
&16X^3 - 16X^2 + X + 1 = 0
\end{aligned}
\tag{2.97}
$$

When solving the cubic, there are three solutions

$$
\begin{aligned}
X &= 0.8357 \\
&= 0.3677 \\
&= -0.2034
\end{aligned}
\tag{2.98}
$$

In the diagram, $\frac{x}{l} = \frac{5}{13} = 0.3846$, and the solution most similar to this is $\frac{x}{l} = 0.3677$. Treating $\frac{x}{l} = 0.3677$ as the solution, in triangle OAP

$$
\begin{aligned}
OP &= X \\
AP &= \frac{1}{2}l = \sqrt{1 - X^2} \\
AO &= 1 \\
R &= \frac{\frac{1}{2}l}{\sqrt{1 - X^2}}
\end{aligned}
\tag{2.99}
$$

Inputting this into the quadratic equation in (2.105) gives

$$
d_2 = 18.3588 \tag{2.100}
$$

This produces an answer similar to the tablet but with more significant figures. However, if we assume the answer is exactly 18, a 5-12-13 triangle can be formed within the given triangle $\triangle ABC$ in Figure 2.58. Where it is assumed that d_2 is as large as possible within the arc ABQ and sits directly in its middle, such that OQ bisects l, the Pythagorean theorem can be applied to an assumed triangle $\triangle OAP$ so that

$$\begin{aligned}
R^2 &= (R - d_2)^2 + \left(\frac{1}{2}\,l\right)^2 \\
R^2 &= 2Rd_2 + {d_2}^2 + \frac{1}{4}\,l^2 \\
2Rd_2 &= {d_2}^2 + \frac{1}{4}\,l^2 \\
{d_2}^2 - 2Rd_2 + \frac{1}{4}\,l^2 &= 0
\end{aligned} \tag{2.101}$$

Where the answer of 18 given on the tablet is assumed for d_2, it is observed that $\triangle OAP$ is a 5-12-13 triangle

$$\begin{aligned}
R &= \frac{{d_2}^2 + \frac{1}{4}\,l^2}{2d} \\
&= \frac{18^2 + 27^2}{2(18)} \\
&= \frac{9^2}{2(18)}(2^2 + 3^2) = \frac{9}{4}(13) \\
OP = R - d_2 &= \frac{9}{4}(13) - 18 \\
&= 9\left(\frac{13}{4} - 2\right) = \frac{9}{4}(5) \\
AP = 27 &= \frac{9}{4}(12)
\end{aligned} \tag{2.102}$$

The appearance of this 5-12-13 triangle, derived by assuming the answer given by the author, could indicate they first constructed this triangle and then created the problem from the solution. However, when the answer is not assumed, the calculations to obtain it (as done previously in this section) are not consistent with this.

Comments

The answer given on the tablet may indicate the author constructed two 5-12-13 triangles to make up the isosceles triangle and worked backwards from the answer to construct the problem. However upon closer examination, the proportions of the triangles are slightly different. Because of this the small circle diameter is 18.3588 rather than 18, making the answer on the tablet correct to two significant figures.

2.8.13.9 Suwa: Seventh Problem

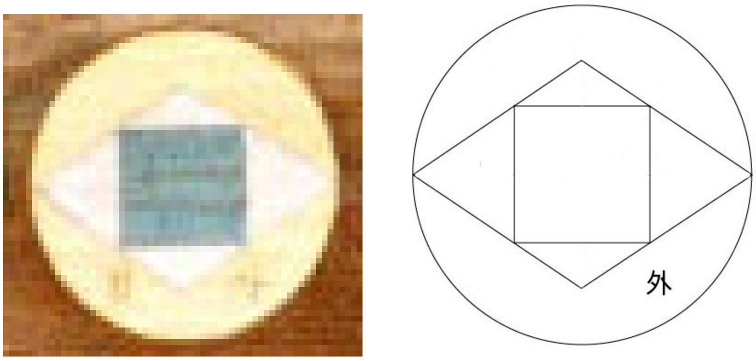


FIGURE 2.60: Left: Seventh Suwa problem. Right: Transcription. (H. Kotera[58]).

Translation

TRANSCRIPTION	TRANSLATION
今有如圖外圓徑内容 菱面内容方面只云長 矢一十八寸短矢八寸問外圓徑幾何	As in the diagram, there is an outer circle (外) which contains a rhombus with a square inside. Say the long sagitta is 18 *sun* and the short sagitta is 8 *sun*. Problem - what is the diameter of the outer circle?
答日　外圓径六十寸	Answer: The diameter of the outer circle is 60 *sun*

Modern Analysis

The two mentioned sagitta in the text are not drawn on the diagram, and there are no arcs on the diagram where the sagitta could lie. However by assuming the sagitta are the altitudes of the two sizes of triangles produced by placing the square inside the rhombus, the answer on the tablet can be obtained. This suggests the term *ya* 矢 which is often translated as sagitta by Mitsuo and Ogawa [54, p. 255] and Komatsu [105, p. 133] can also be applied to lengths other than the distance from the centre of an arc to the centre of the base.

In Figure 2.61, let the side length of the given rhombus be EC. Assume half the side of the square is AB, EF is the short sagitta, and BC the long sagitta. Assume two right angle triangles $\triangle AEF$ and $\triangle ABC$. Say the outer circle diameter is d. In Figure 2.61 the two triangles $\triangle ABC$ and $\triangle AEF$ are similar. Since $AF = AB$, they can be

[58] See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

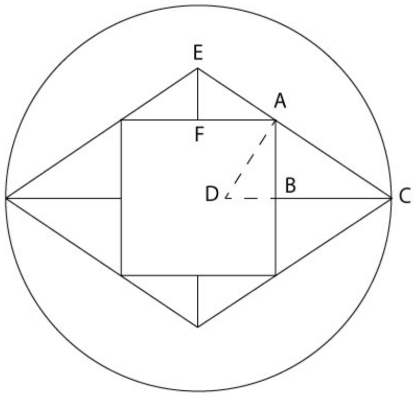


FIGURE 2.61: Suwa: Seventh problem analysis

combined to form triangle $\triangle ADC$. The right angle triangle altitude theorem can be applied to obtain the value of AB

$$
\begin{aligned}
&ABD = AEF \\
&DB = EF \\
&AB = \sqrt{DB \cdot BC} \\
&AB = 12 && (2.103)
\end{aligned}
$$

The value of d can be found by combining the square side length with twice the long sagitta

$$
\begin{aligned}
&d = 2BC + 2AB \\
&d = 2(18) + 2(12) \\
&d = 60 && (2.104)
\end{aligned}
$$

Comments

The omission of the sagitta from the diagram means additional contemplation. Perhaps even trial and error with potential locations is required. Since there are no arcs in the diagram, the term *ya* 矢 must also have a meaning broader than just sagitta, for it is seen to incorporate altitudes of triangles as well as standard sagitta.

2.8.13.10 Suwa: Eighth Problem

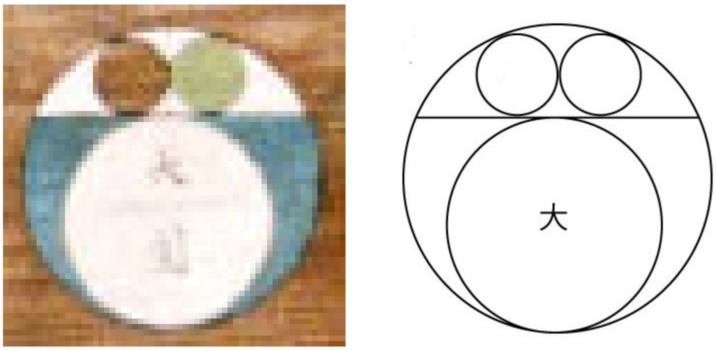


FIGURE 2.62: Left: Eighth Suwa problem. Right: Transcription. (H. Kotera[59]).

Translation

TRANSCRIPTION	TRANSLATION
今有如圖外圓径内容大小圓径三個只云大圓径三十六寸小圓径二十四寸問外圓径幾何	As in the diagram, there is an outer circle (外) which contains three circles, large (大) and small (小). Say the large circle diameter is 36 *sun*, and the diameter of the small circle is 24 *sun*. Problem - what is the diameter of the outer circle?
答日　外圓径六十四寸	Answer: The diameter of the outer circle is 64 *sun*

Modern Analysis

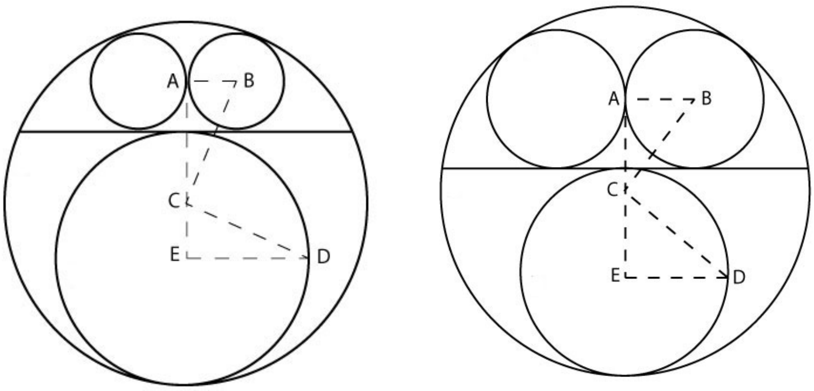


FIGURE 2.63: Suwa: Eighth problem analysis. Left: Figure proportions as per tablet diagram. Right: Figure proportions as per given values.

[59]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

In Figure 2.63, a transcription of the diagram appears on the left and on the right is the diagram produced in Illustrator using values from the text. Though only the large circle is labelled by the author, the outer circle can be assumed to be a, the large b, and the smaller c. In this instance a general method can be applied to both problems without alternatives required.

In Figure 2.63, assume two right angle triangles $\triangle ABC$ and $\triangle CDE$ formed by connecting the small and large circles with the diameter of the outer. Let the diameter of the outer circle be d_1, the diameter of the big circle be d_2 and the diameter of the small circles be d_3.

In triangle $\triangle ABC$

$$AB = \frac{d_3}{2}$$
$$BC = \frac{d_1}{2} - \frac{d_3}{2}$$
$$AC = \sqrt{\left(\frac{d_1}{2} - \frac{d_3}{2}\right)^2 - \left(\frac{d_3}{2}\right)^2} \tag{2.105}$$

In triangle $\triangle CDE$

$$DE = \frac{d_2}{2}$$
$$DC = \sqrt{\left(\frac{d_1}{2} - \frac{d_2}{2}\right)^2 + \left(\frac{d_2}{2}\right)^2}$$
$$CE = \frac{d_1}{2} - \frac{d_2}{2} \tag{2.106}$$

From Figure 2.63

$$AE = AC + CE$$
$$AE = \frac{d_2}{2} + \frac{d_3}{2}$$
$$\frac{d_2}{2} + \frac{d_3}{2} = \sqrt{\left(\frac{d_1}{2} - \frac{d_3}{2}\right)^2 - \left(\frac{d_3}{2}\right)^2} + \frac{d_1}{2} - \frac{d_2}{2}$$

$$\frac{d_2}{2} + \frac{d_3}{2} = \sqrt{\left(\frac{d_1}{2} - \frac{d_3}{2}\right)^2 - \left(\frac{d_3}{2}\right)^2} + \frac{d_1}{2} - \frac{d_2}{2}$$
$$d_2 + d_3 = \sqrt{(d_1 - d_3)^2 - (d_3)^2} + d_1 - d_2$$
$$2d_2 + d_3 = \sqrt{d_1(d_1 - 2d_3)} + d_1$$
$$2d_2 + d_3 - d_1 = \sqrt{d_1(d_1 - 2d_3)} \tag{2.107}$$

Square both sides and solving for d_1

$$(2d_2 + d_3 - d_1)^2 = d_1(d_1 - 2d_3)$$
$$(2d_2 + d_3 - d_1)^2 = {d_1}^2 - 2d_1d_3$$
$$4{d_2}^2 + 4d_1d_3 + {d_3}^2 - 4d_1d_2 = 0$$
$$(2d_2 + d_3)^2 = 4d_1d_2$$
$$d_1 = \frac{(2d_2 + d_3)^2}{4d_2} = 64 \tag{2.108}$$

Comments

Because a general method can be used for both diagrams without alteration of the positions of the auxillary triangles used in the solution, this problem induces contemplation of general principles that can be applied to a wide variety of similar diagrams.

2.8.13.11 Suwa: Ninth Problem

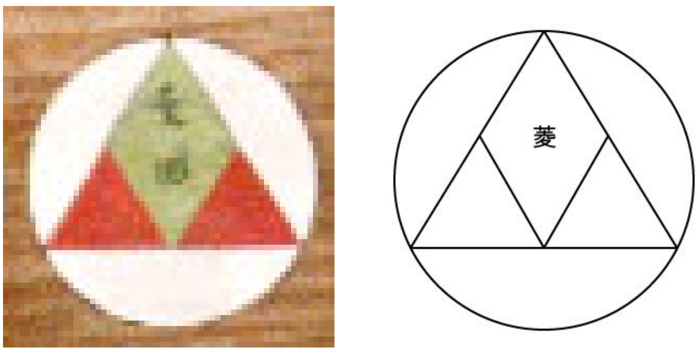


FIGURE 2.64: Left: Ninth Suwa problem. Right: Transcription. (H. Kotera[60]).

[60]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

Translation

Transcription	Translation
今有如圖外圓径内容三角面亦云三角面内容菱面只云外圓六十寸問菱面幾何	As in the diagram, there is an outer circle (外) which contains an equilateral triangle. Also say inside the triangle there is a rhombus. Say the diameter of the outer circle is 60 *sun*. Problem - what are sides of the rhombus?
答曰　菱面二十五寸九分八	Answer: The sides of the rhombus are 25 *sun* 9 *bu* 8

Modern Analysis

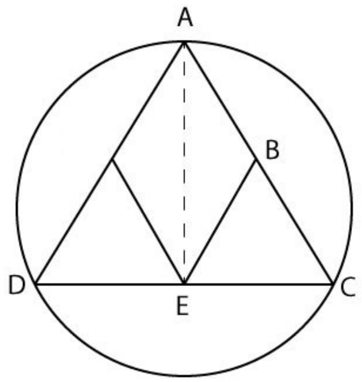


Figure 2.65: Suwa: Ninth problem analysis

In Figure 2.65, let the given outer circle have a radius R. Let the equilateral triangle be $\triangle ABC$ with an altitude of AE. Let AB be the side length of the rhombus. To find the side length AB, the altitude AE is first obtained from treating a as the circumscribing circle of $\triangle ACD$ such that

$$R = \frac{2AE}{3} \tag{2.109}$$

From this AB can be obtained by finding the side length AC of $\triangle ACD$ and then dividing by half to find AB

$$R = \frac{2AE}{3}$$

$$3R = 2AE$$

$$\frac{3R}{2} = AE$$

$$AC = AE \div \frac{\sqrt{3}}{2}$$

$$AB = AC \div 2$$

$$AB = 25.98076211 \tag{2.110}$$

2.8.13.12 Suwa: Tenth Problem

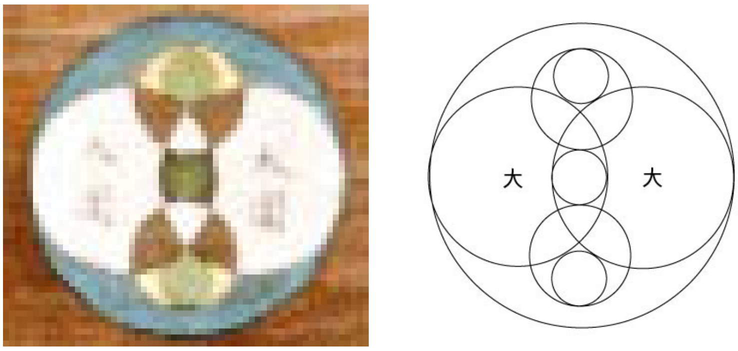


FIGURE 2.66: Left: Tenth Suwa problem. Right: Transcription. (H. Kotera[61]).

Translation

TRANSCRIPTION	TRANSLATION
今有如圖外圓径内容大圓中圓小圓只云中圓径一十寸小圓径五寸問外圓径幾何	As in the diagram, there is an outer circle (外) which contains large (大), medium (中), and small circles (小). Say the diameter of the medium circles is 10 *sun* and the diameter of the small circles is 5 *sun*. Problem - what is the diameter of the outer circle?
答日　外圓径三十五寸	Answer: The diameter of the outer circle is 35 *sun*

Modern Analysis

In Figure 2.63, let the radius of the given outer circle be R, the radius of the large circles be r_1, the radius of the medium circles be r_2, and the radius of the small circles be r_3.

[61]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

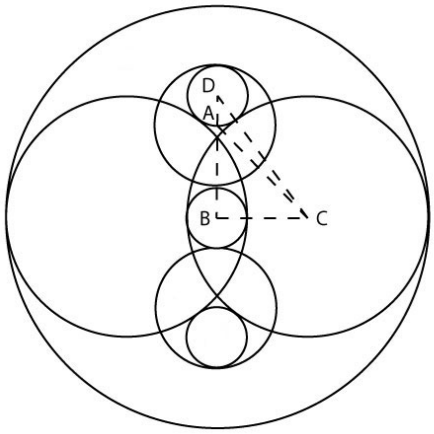


FIGURE 2.67: Suwa: Tenth problem analysis

Assume triangles $\triangle ABC$ and $\triangle BCD$ which connect the centres of the circles. In triangle $\triangle BCD$

$$\begin{aligned} CD &= r_1 + r_3 \\ BC &= r_1 - r_3 \\ BD &= 2r_2 = 4r_2 \end{aligned} \tag{2.111}$$

Applying the Pythagorean theorem

$$\begin{aligned} &CD^2 = BC^2 + BD^2 \\ &(r_1 + r_3)^2 = (r_1 - r_3)^2 + (4r_3)^2 \\ &{r_1}^2 + 2r_1r_3 + {r_3}^2 = {r_1}^2 - 2r_1r_3 + {r_3}^2 + 16r_3 \\ &4r_1r_3 = 16r_3 \\ &r_1 = 4r_3 \\ &2r_1 = 8r_3 \end{aligned} \tag{2.112}$$

From Figure 2.67 and 2.112

$$\begin{aligned} 2R &= 2r_1 + (2r_1 - 2r_3) \\ &= 4r_1 - 2r_3 \\ &= 2(8r_3) - 2r_3 \\ &= 16r_3 - 2r_3 \\ &= 14r_3 = 35 \end{aligned} \tag{2.113}$$

2.8.13.13 Suwa: Eleventh Problem

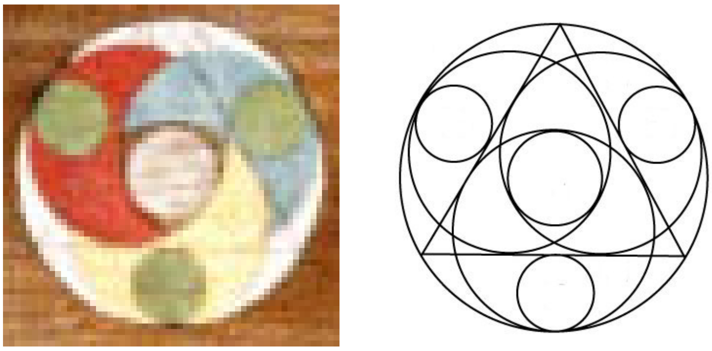


FIGURE 2.68: Left: Eleventh Suwa problem. Right: Transcription. (H. Kotera[62]).

Translation

TRANSCRIPTION	TRANSLATION
今有如圖外圓径内容甲乙丙圓径只云甲圓径一十四寸問乙圓径幾何	As in the diagram, there is an outer circle (外) which contains circles *kō* 甲, *otsu* 乙, and *hei* 丙. Say the diameter of circle *kō* 甲 is 14 *sun*. Problem - what is the diameter of circle *otsu* 乙?
答日　乙圓径一十寸〇一厘	Answer: The diameter of circle *otsu* 乙 is 10 *sun* 01 *rin*

Modern Analysis

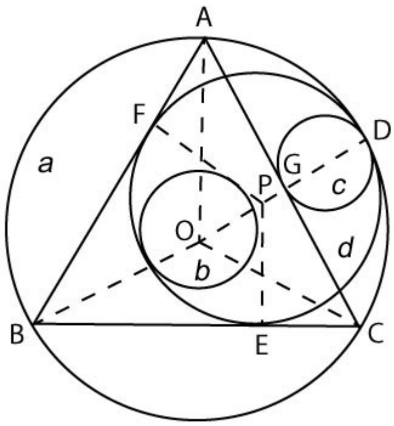


FIGURE 2.69: Suwa: Eleventh problem analysis

[62]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

In Figure 2.68, there are no labels provided to indicate which circles are which in the diagram. By observing the diagram it can be seen that given the numerical value of *kō* 甲 as 14 and *otsu* 乙 as 10.01 the most likely candidates for these circles are the innermost lying in the intersection of the three largest for *kō* 甲 and the smallest lying in the arcs made by the triangle as *otsu* 乙. For, if *otsu* 乙 were the innermost circle and *kō* 甲 the largest of the internal circles, it would visibly be well over half the radius of the large circles.

Given this, as in Figure 2.69 assume the outer circle is a, circle *kō* 甲 is b, circle *otsu* 乙 is c, and circle *hei* 丙 is d. Let the radius of the outer circle be R, the radius of *kō* 甲 be r_1, the radius of *otsu* 乙 be r_2, and the radius of *hei* 丙 be r_3. Assume equilateral triangles $\triangle ABC$, $\triangle OCD$, and $\triangle OAD$, and right angle triangles $\triangle PBE$ and $\triangle PBF$.

In Figure 2.69, since the line AC bisects the line OD

$$
\begin{aligned}
OD &= R = OG + GD \\
OG &= GD = 2r_2 \\
2r_2 &= \frac{1}{2}R \qquad (2.114)
\end{aligned}
$$

Because the right angle triangle $\triangle PBE$ is 30:60:90

$$
\begin{aligned}
BP &= BD - r_3 \\
BD &= 2R \\
PE &= r_3 \qquad (2.115) \\
\frac{r_3}{2R - r_3} &= \frac{PE}{BP} = \frac{1}{2} \\
2r_3 &= 2R - r_3 \\
2R &= 3r_3 \\
R &= \frac{3}{2}r_3 \qquad (2.116)
\end{aligned}
$$

From (2.114) and (2.115), r_2 can be found in terms of r_3 as follows

$$
\begin{aligned}
2r_2 &= \frac{1}{4}2R \\
r_3 &= \frac{1}{3}2R \\
2r_2 &= \frac{3}{4}r_3 \qquad (2.117)
\end{aligned}
$$

From triangle OAG and (2.115)

$$
\begin{aligned}
&OA = R = \frac{3}{2}r_3\\
&R = 2r_3 - r_1\\
&R - 2r_3 = -r_1\\
&\frac{3}{2}r_3 - 2r_3 = -r_1\\
&-\frac{r_3}{2} = -r_1\\
&2r_1 = r_3 \qquad (2.118)
\end{aligned}
$$

From (2.117) and (2.118)

$$
\begin{aligned}
2r_2 &= \frac{3}{4}r_3\\
&= \frac{3}{4}2r_1\\
&= \frac{3}{4}14 = 10.5 \qquad (2.119)
\end{aligned}
$$

The answer obtained is slightly different to the tablet, indicating the tablet is correct to two significant figures.

Comments

Since the original diagram on the *sangaku* does not label the circles, it is from observation and the values given in the text that which circle is which is established. A great deal of contemplation regarding the figures on the diagram and the values in the text is required, and the observer is not able to rely on knowledge of the ranking system of the Chinese calender (mentioned in section 2.10) alone to connect the labels with the circles by virtue of their size in the diagram. It is instead from the geometry of the figure that the labels are ascertained. The figures are also not drawn accurately, as given the diameter of the circle *hei* 丙 is 28 the diameter of *kō* 甲 should be equal to the radius r_3. However, *kō* 甲 is drawn smaller than the radius of *hei* 丙. This makes it more difficult to determine which circle is which.

2.8.13.14 Suwa: Twelfth Problem

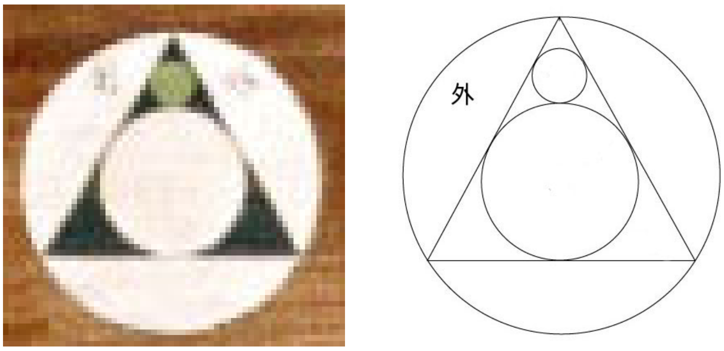


FIGURE 2.70: Left: Twelfth Suwa problem. Right: Transcription. (H. Kotera[63]).

Translation

TRANSCRIPTION	TRANSLATION
今有如圖外圓径内容 三角面亦云三角内容 大小圓径二個只云 外圓径五十寸問 小圓径幾何	As in the diagram, there is an outer circle (外) which contains an equilateral triangle. Also say inside the triangle there are 2 circles large (大) and small (小). Say the outer circle diameter is 50 *sun*. Problem - what is the diameter of the small circle?
答日　小圓径八寸三分三	Answer: The diameter of the small circle is 8 *sun* 3 *bu* 3

Modern Analysis

In Figure 2.71, let the centre of the given outer circle be A, the centre of the large circle also be A, and the centre of the small circle be B. Say the equilateral triangle is $\triangle XYZ$. Assume two 30-60-90 triangles $\triangle ACX$ and $\triangle ABE$. From Figure 2.71

$$\begin{aligned} AX &= R \\ AC &= r_1 \\ BD &= r_2 \end{aligned} \tag{2.120}$$

[63]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

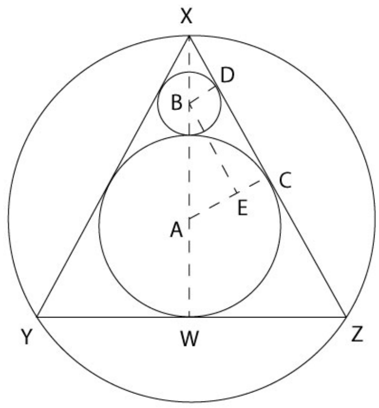


FIGURE 2.71: Suwa: Twelfth problem analysis

From $\triangle ACX$

$$\begin{aligned} r_1 &= AC \\ &= \frac{1}{2}AX \\ &= \frac{1}{2}R \end{aligned} \tag{2.121}$$

Then from $\triangle ABE$ the value of r_2 can be obtained

$$\begin{aligned} &AB = 2AE \\ &AB = r_1 + r_2 \\ &AE = r_1 - r_2 \\ &r_1 + r_2 = 2(r_1 - r_2) \\ &3r_2 = r_1 \\ &r_2 = \frac{1}{3}r_1 \\ &2r_2 = 8.333333333 \end{aligned} \tag{2.122}$$

2.8.13.15 Suwa: Thirteenth Problem

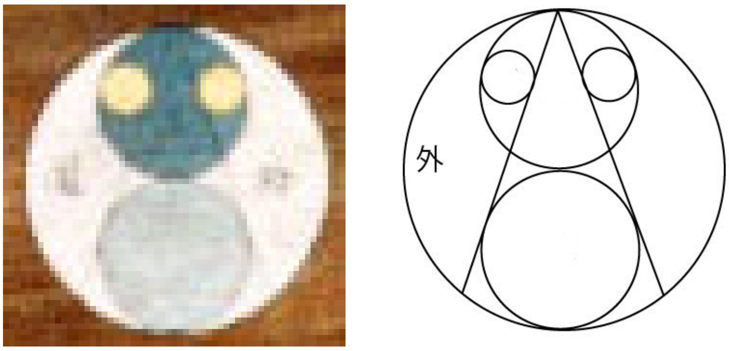


FIGURE 2.72: Left: Thirteenth Suwa problem. Right: Transcription. (H. Kotera[64]).

Translation

TRANSCRIPTION	TRANSLATION
今有如圖外圓径内容大小圓径只云大圓径三十六寸問小圓径幾何	As in the diagram, there is an outer circle (外) which contains large (大) and small circles (小). Say the diameter of the large circle is 36 *sun*. Problem - what is the diameter of the small circle?
答日　小圓径一十二寸	Answer: The diameter of the small circle is 12 *sun*

Modern Analysis

In Figure 2.72 assume the outer circle is a, the large circles are b, and the small circles are c. Let the radius of the large circles be R and the radius of the small circles be r_1. Assume triangles $\triangle ABC$, $\triangle ADE$, $\triangle AOP$ and line PQ. In Figure 2.73, in the right angle triangles $\triangle ADE$ and $\triangle AOP$

$$\begin{aligned} AD &= 3R \\ DE &= R \\ AO &= R \end{aligned} \tag{2.123}$$

[64]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

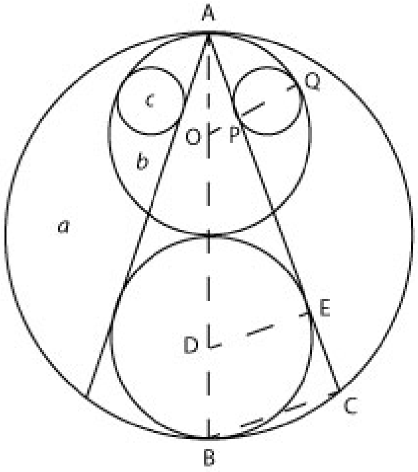


FIGURE 2.73: Suwa: Thirteenth problem analysis

The triangles $\triangle ADE$ and $\triangle AOP$ are similar, with

$$\begin{aligned}AO &= \frac{1}{3}AD \\ OP &= \frac{1}{3}DE = \frac{R}{3}\end{aligned} \tag{2.124}$$

Having determined the value of OP, the diameter of the small circles c can be found as follows

$$\begin{aligned}2r_1 &= PQ \\ &= OQ - OP \\ &= R - \frac{R}{3} \\ &= \frac{2}{3}R \\ &= \frac{1}{3}2R = 12\end{aligned} \tag{2.125}$$

2.8.13.16 Suwa: Fourteenth Problem

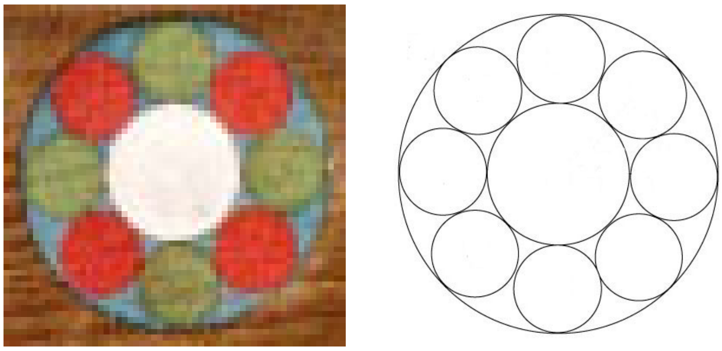


FIGURE 2.74: Left: Fourteenth Suwa problem. Right: Transcription. (H. Kotera[65]).

Translation

TRANSCRIPTION	TRANSLATION
今有如圖外圓径内容甲乙圓径九個只云外圓径五十寸問乙圓径幾何	As in the diagram, there is an outer circle (外) which contains 9 circles *kō* 甲 and *otsu* 乙. Say the diameter of the outer circle is 50 *sun*. Problem - what is the diameter of circle *otsu* 乙?
答日　外圓径三十五寸	Answer: The diameter of circle *otsu* 乙 is 22 *sun* 3 *bu* 25

Modern Analysis

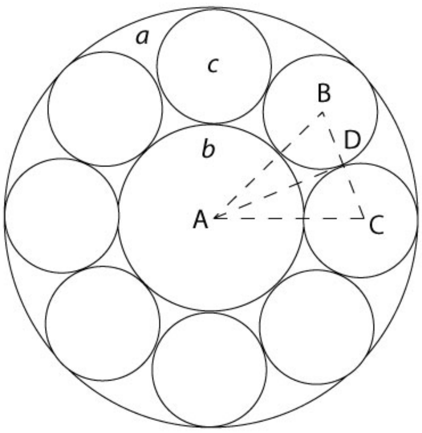


FIGURE 2.75: Suwa: Fourteenth problem analysis

[65]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

In Figure 2.74, observing the values associated with the circles in the text, assume *otsu* 乙 is the innermost circle and the 8 small circles are *kō* 甲. Let the radius of the outer circle be R, the radius of circle *otsu* 乙 be r_1, and the radius of the circles *kō* 甲 be r_2. Assume there are also triangles $\triangle ABC$, $\triangle ABD$, and $\triangle ACD$.

From Figure 2.75, since there are 8 circles c contained within the larger circle a it can be determined that the triangles $\triangle ABD$ and $\triangle ACD$ are 22.5-67.5-90. This is because $360 \div 8 = 45$, making the angles at A in $\triangle ABC$ 45 degrees. Given this

$$R = r_1 + 2r_2$$

$$\sin 22.5 = \frac{r_2}{r_1 + r_2}$$

$$r_2 = \frac{r_1 \sin 22.5}{1 - \sin 22.5} \tag{2.126}$$

From (2.126)

$$R = r_1 + 2\left(\frac{r_1 \sin 22.5}{1 - \sin 22.5}\right)$$

$$r_1 = \frac{1 - \sin 22.5}{1 + \sin 22.5} \cdot R$$

$$2r_1 = 22.323 \tag{2.127}$$

Comments

As with Problem 11, the usual manner of labelling circles with the Chinese calender in terms of their size - with *kō* 甲 assigned to the largest circle, *otsu* 乙 to the next largest, and so on - does not apply in this problem. The only way to determine which circle is which is by observation of their sizes and the values given in the text.

2.8.13.17 Suwa: Fifteenth Problem

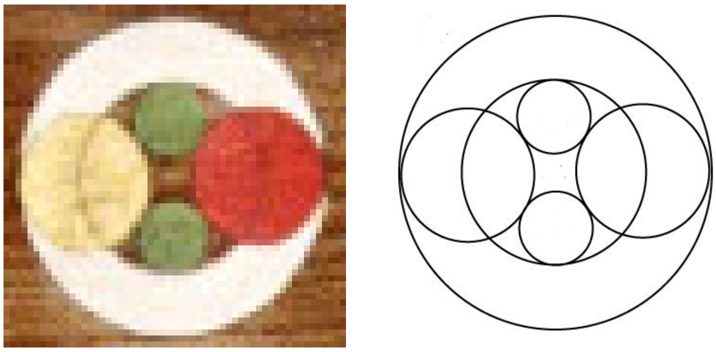


FIGURE 2.76: Left: Fifteenth Suwa problem. Right: Transcription. (H. Kotera[66]).

Translation

TRANSCRIPTION	TRANSLATION
今有如圖外圓径内容 大中小圓径只云小圓径 九十寸問中圓径幾何	As in the diagram, there is an outer circle (外) which contains large (大), medium (中) and small circles (小). Say the diameter of the small circles is 90 *sun*. Problem - what is the diameter of the medium circle(s)?
答日　中圓径二百一十寸	Answer: The diameter of the medium circle(s) is 210 *sun*

Modern Analysis

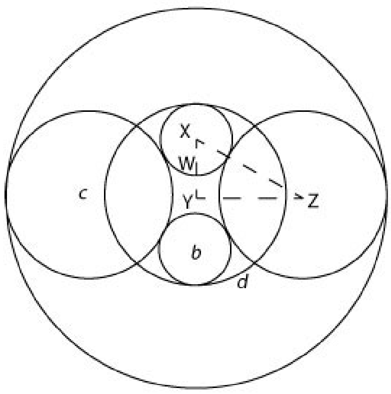


FIGURE 2.77: Suwa: Sixteenth problem analysis

[66]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

This problem is underdetermined, as in the diagram in Figure 2.76 none of the circles are labelled by the author and there is not enough information given in the text to find the solution.

If one assumes as in Figure 2.77 that the small circles are the two inner circles b, and the medium circles the two circles c, then the answer obtained on the tablet cannot be produced. If a triangle $\triangle XYX$ is assumed as in Figure 2.77, where half the inner gap between the small circles WY is labelled a, then the following calculation can be made

$$
\begin{aligned}
&WY = a\\
&XZ = b + c\\
&XY = b + a\\
&YZ = a + c\\
&(b + c)^2 = (b + a)^2 + (a + c)^2\\
&b^2 + 2bc + c^2 = b^2 + 2ab + a^2 + c^2 + 2ac + WY^2\\
&bc = a^2 + ab + ac\\
&c = \frac{b + a}{b - a} \cdot a
\end{aligned} \tag{2.128}
$$

This calculation relies on knowledge of the space a, which is not provided on the tablet. If it is assumed that the circles c lie on the circumference of the large circle d, then a calculation which does not require a can be produced

$$
\begin{aligned}
&a + 2b = a + c\\
&c = 2b = 180
\end{aligned} \tag{2.129}
$$

However this calculation gives 180 as the answer instead of the 210 on the tablet. By assuming that the middle circle referred to in the text is instead the large circle d, also assuming that the circles c lie on its circumference, we can calculate

$$
\begin{aligned}
&2d = 210\\
&2b = 90\\
&a = d - 2b = 105 - 90 = 15\\
&(b + c)^2 = (b + a)^2 + (a + c)^2\\
&(45 + c)^2 = 60^2 + (15 + c)^2\\
&60c = 60^2 + 15^2 - 45^2 = 15^2 \cdot 8\\
&c = 30
\end{aligned} \tag{2.130}
$$

This also produces a value for c that does not fit with the text. With these options exhausted, it appears that the problem is underdetermined to the extent that the answer on the tablet cannot be derived.

2.8.13.18 Suwa: Sixteenth Problem

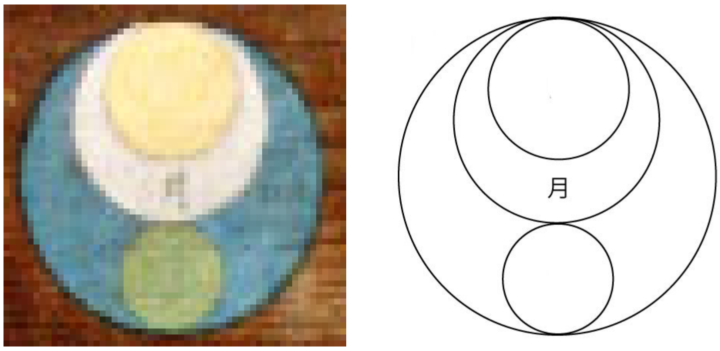


FIGURE 2.78: Left: Sixteenth Suwa problem. Right: Transcription. (H. Kotera[67]).

Translation

Transcription	Translation
今有如圖外圓径内容 三個只云月星和四十 二寸問日圓径幾何	As in the diagram, there is an outer circle (外) which contains three circles. Say the circles *tsuki* 月 and *hoshi* 星 combined are 42 *sun*. Problem - what is the diameter of circle *hi* 日?
答日　日圓径二十一寸	Answer: The diameter of circle *hi* 日 is 21 *sun*

Modern Analysis

In Figure 2.78, only the circle *tsuki* 月 is labelled. The naming of the circles differs to other problems, using neither the size descriptive labels or Chinese calender labels. Instead the words *tsuki* 月 'moon', *hi* 日 'sun', and *hoshi* 星 'star' appear. To determine which circles the labels *hi* 日 and *hoshi* 星 apply to, the values in the text and the diagram must be analysized. The text gives the diameter of the circle *hi* 日 as half the sum of *tsuki* 月 and *hoshi* 星. Since the smallest circle in the diagram is less than the diameter of the uppermost, it would be too small to amount to half the combined value of the uppermost with *tsuki* 月. Since the uppermost is larger than the smallest, it is a more likely candidate for the value *hi* 日. Given this, as in Figure 2.79 assume that circle *hoshi* 星 is a, circle *tsuki* 月 is b, and circle *hi* 日 is c.

[67]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

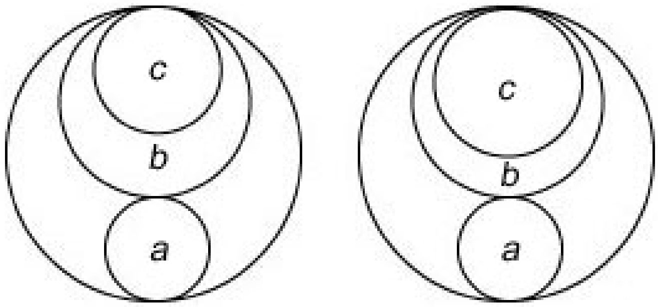


FIGURE 2.79: Suwa: Sixteenth problem analysis. Left: Figure proportions as per tablet diagram. Right: Figure proportions as per given values.

When examining the text further, it appears not enough information is given to find the solution provided, suggesting that there is an implicit assumption. One possibility is that it is assumed that the diameters of the circles are in an arithmetic progression. Where r_1 is the radius of *hoshi* 星, r_2 the radius of *hi* 日, r_3 the radius of *tsuki* 月, and d the constant difference between them, this would give

$$r_1 \qquad r_2 = r_1 + d \qquad r_3 = r_1 + 2d \tag{2.131}$$

Employing this method, r_2 can be found without needing to find the individual values of r_1 and r_3 as follows

$$\begin{aligned} 21 = r_1 + r_3 &= 21 \\ &= r_1 + (r_1 + 2d) \\ &= 2r_1 + 2d \\ &= 2(r_1 + d) \\ &= 2r_2 \end{aligned} \tag{2.132}$$

Comments

This problem requires analysis of the text and diagram to make an assumption regarding which circles are which in the diagram. Additional assumptions are also required due to the given information being insufficient on its own to produce the answer. This underdetermination means there will always be an aspect of uncertainty. For even if using the potential solution given in the modern analysis section - which assumes the figures are in an arithmetic progression - there is no way to prove such a progression holds since the individual diameters of the two circles whose sum is 42 is unknown and cannot be determined using the details in the text.

This underdetermination suggests that the act of contemplating the problem may have been more important to the author than the solving of it. For while presented in the typical style of a problem, it does not fulfill the requirements due to insufficient information. This may mean it should be treated as something slightly different, and a work in which mathematical shapes and language is admired and investigated rather than a problem solved.

2.8.13.19 Suwa: Seventeenth Problem

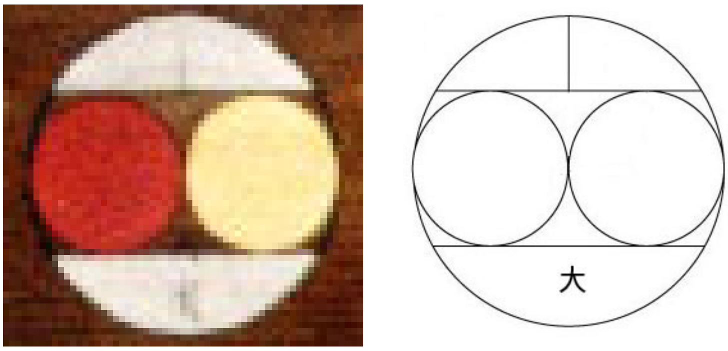


FIGURE 2.80: Left: Seventeenth Suwa problem. Right: Transcription. (H. Kotera[68]).

Translation

TRANSCRIPTION	TRANSLATION
今有如圖大圓径内容径 中圓二個只云大圓径 四十寸矢一十寸問 中圓径幾何	As in the diagram, there is a large circle (大) which contains two medium circles (中). Say the diameter of the large circle is 40 *sun* and the sagitta is 10 *sun*. Problem - what is the diameter of the medium circles?
答日　中圓径二十寸	Answer: The diameter of the medium circles is 20 *sun*

Modern Analysis

In Figure 2.81, the left-hand side shows the transcribed diagram, and the right a version of the problem with different proportions. Let the radius of the given outer circle be R, the sagitta length be s, and the diameter of the medium circles be d_1. Assume a triangle $\triangle ABC$. It can be visually seen in the left diagram that the radius of the medium circles is the radius of the outer circle minus the sagitta - making it $20 - 10 = 10$.

[68]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

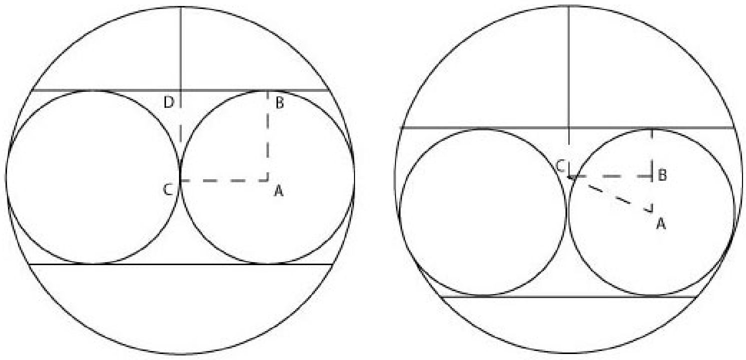


FIGURE 2.81: Suwa: Seventeenth problem analysis. Left: Figure proportions as per tablet diagram. Right: Figure proportions as per given values.

Because previous problems have produced general methods that can be applied to a variety of different diagrams with a similar configuration and geometry, a general method is sought this problem. To find a general method, the diagram on the right in Figure 2.81 can be used.

From $\triangle ABC$

$$
\begin{aligned}
AB &= \frac{d_1}{2} + s - R \\
BC &= \frac{d_1}{2} \\
AC &= R - \frac{d_1}{2}
\end{aligned} \tag{2.133}
$$

Applying the Pythagorean theorem to $\triangle ABC$

$$
\begin{aligned}
&AB^2 + BC^2 = AC^2 \\
&\left(R - \frac{d_1}{2}\right)^2 = \left(\frac{d_1}{2} + s - R\right)^2 + \left(\frac{d_1}{2}\right)^2 \\
&\left(R - \frac{d_1}{2}\right)^2 = \frac{{d_1}^2}{2} + d_1(s-R) + R^2 - 2Rs + s \\
&-\frac{{d_1}^2}{4} - d_1 s + 2Rs - s^2 = 0
\end{aligned}
$$

$$
\begin{aligned}
&{d_1}^2 + 4d_1 s = 4(2Rs - s^2) \\
&(d_1 + 2s)^2 = 4s^2 + 4(2Rs - s^2) \\
&d_1 = 2\sqrt{2Rs} - 2s \quad \text{or} - 2s - \sqrt{2Rs}
\end{aligned} \tag{2.134}
$$

Since the second equation produces a negative value and the answer on the tablet is positive, the first equation is adopted

$$\begin{aligned}
d_1 &= 2\sqrt{2Rs} - 2s \\
d_1 &= 2\sqrt{2 \cdot 20 \cdot 10} - 2 \cdot 10 \\
d_1 &= 20
\end{aligned} \tag{2.135}$$

When applying this method to the left-hand diagram, the triangle $\triangle ABC$ cannot be formed, but it can be observed that

$$\begin{aligned}
AC &= R - \frac{d_1}{2} \\
AC &= \frac{d_1}{2}
\end{aligned} \tag{2.136}$$

From this equality the answer can be found

$$\begin{aligned}
\frac{d_1}{2} &= R - \frac{d_1}{2} \\
d_1 &= R \\
d_1 &= 20
\end{aligned} \tag{2.137}$$

Using this method, when the diagram is presented such as on the tablet, or with slightly differently sized circles and sagitta, the answer can be found by creating the triangle $\triangle ABC$.

2.8.13.20 Suwa: Eighteenth Problem

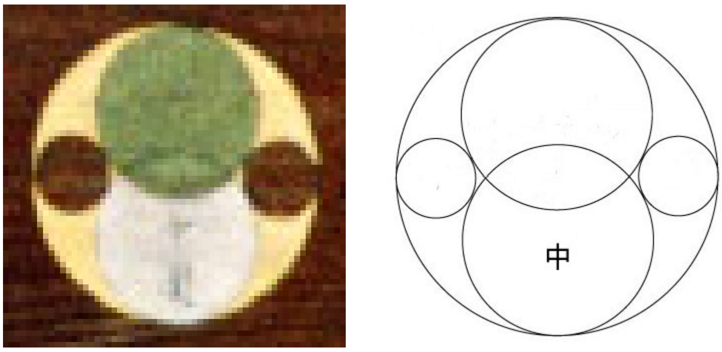


FIGURE 2.82: Left: Eighteenth Suwa problem. Right: Transcription. (H. Kotera[69]).

Translation

TRANSCRIPTION	TRANSLATION
今有如圖外圓径内容中圓径二個小圓径二個只云外圓径五十寸中圓径三十寸問小圓径幾何	As in the diagram, there is an outer circle (外) which contains two medium circles (中) and two small circles (小). Say the diameter of the outer circle is 50 *sun* and the diameter of the medium circles is 30 *sun*. Problem - what is the diameter of the small circles?
答曰　小圓径一十二寸五分	Answer: The diameter of the small circle is 12 *sun* 5 *bu*

Modern Analysis

In Figure 2.83 let the outer circle have a radius R, the medium circles have a radius r_1, and the small circles have a radius r_2. Assume a right angle triangle $\triangle ABC$.

In triangle $\triangle ABC$

$$
\begin{aligned}
AB &= R - r_1 \\
AC &= R - r_2 \\
BC &= r_1 + r_2
\end{aligned} \tag{2.138}
$$

[69]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

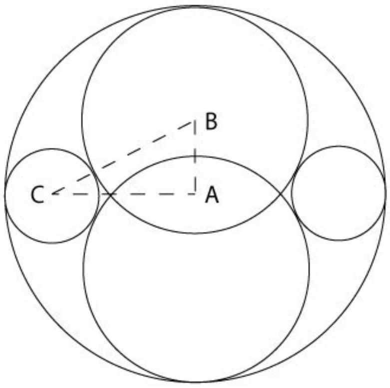


FIGURE 2.83: Suwa: Eighteenth problem analysis

Applying the Pythagorean theorem and solving for r_2

$$BC^2 = AB^2 + AC^2$$
$$(r_1 + r_2)^2 = (R - r_1)^2 + (R - r_2)^2 \tag{2.139}$$
$$(25 - r_2)^2 + 10^2 = (15 + r_2)^2$$
$$10^2 + 25^2 - 50r_2 + {r_2}^2 = 15^2 + 30r_2 + {r_2}^2$$
$$725 - 225 = 30r_2 + 50r_2$$
$$500 = 80r_2$$
$$r_2 = 6.25$$
$$2r_2 = 12.5 \tag{2.140}$$

2.8.13.21 Suwa: Ninteenth Problem

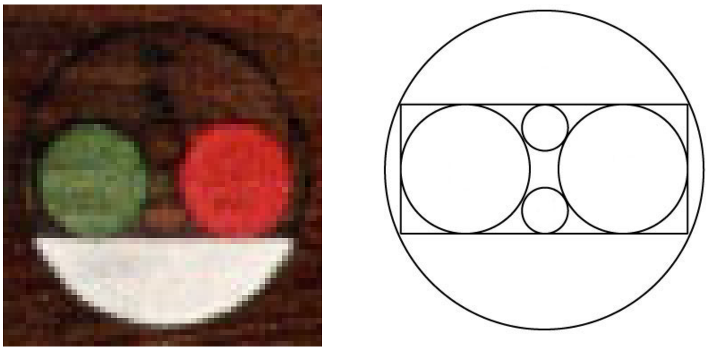


FIGURE 2.84: Left: Nineteenth Suwa problem. Right: Transcription. (H. Kotera[70]).

[70]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

Translation

TRANSCRIPTION	TRANSLATION
今有如圖外圓径内直容四個只云甲圓径一十五寸乙圓径一十一寸問外圓径幾何	As in the diagram, there is an outer circle (外) which contains four lines. Say the diameter of circle *kō* 甲 is 15 *sun* and the diameter of circle *otsu* 乙 is 11 *sun*. Problem - what is the diameter of the outer circle?
答日　外圓径四十三寸五八	Answer: The diameter of the outer circle is 43 *sun* 58

Modern Analysis

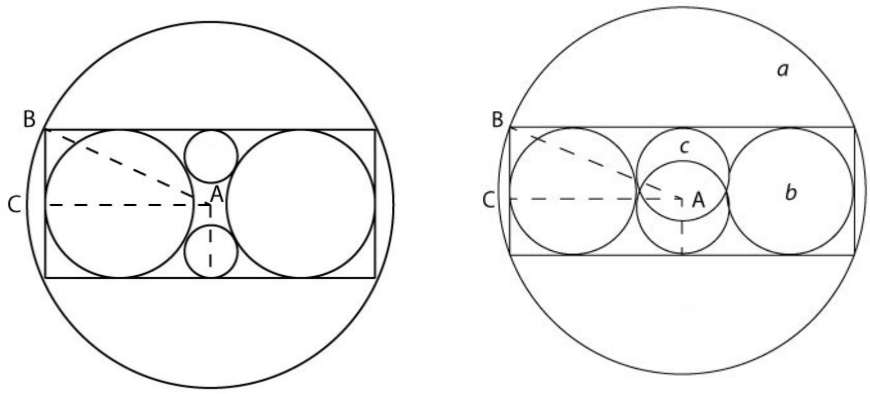


FIGURE 2.85: Suwa: Nineteenth problem analysis. Left: Figure proportions as per tablet diagram. Right: Figure proportions as per given values.

In Figure 2.85, the diagram on the left represents the configuration given on the tablet, and the right that produced in Illustrator. In 2.84, from examining the values of the circles *kō* 甲 and *otsu* 乙, assume *kō* 甲 represents the larger internal circles and *otsu* 乙 the smaller. Although the sizing of the circles differs, the same general method can be found and applied to both diagrams. Let the radius of the outer circle be R, the diameter of the circle *kō* 甲 be d_1, and the diameter of the circle *otsu* 乙 be d_2. Assume there be a right angle triangle $\triangle ABC$.

From $\triangle ABC$

$$BC = \frac{d_1}{2}$$

$$AC = \sqrt{d_1 d_2} + \frac{d_1}{2}$$

$$AB = R$$

Applying the Pythagorean theorem and solving for R

$$AC^2 = BC^2 + AB^2$$
$$AC^2 = \left(\sqrt{d_1 d_2} + \frac{d_1}{2}\right)^2 + \left(\frac{d_1}{2}\right)^2$$
$$AC = 21.68359953$$
$$2R = 43.36719906 \tag{2.141}$$

This answer differs to the one on the tablet, indicating the tablet is correct to two significant figures. This method is able to be applied to both diagrams due even though the circle sizes differ.

2.8.13.22 Suwa: Twentieth Problem

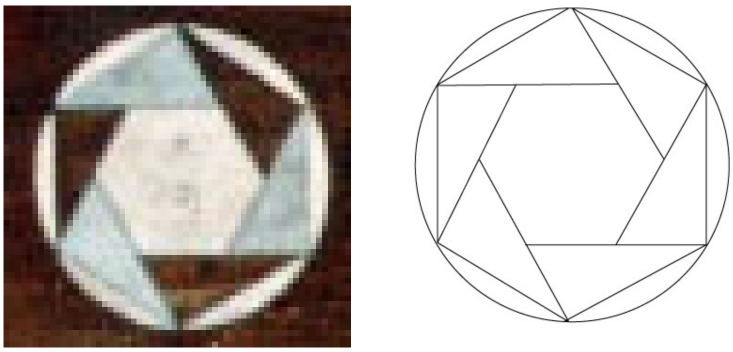


FIGURE 2.86: Left: Twentieth Suwa problem. Right: Transcription. (H. Kotera[71]).

Translation

TRANSCRIPTION	TRANSLATION
今有如圖外圓径内容六角面 只云六角面矢二十四寸矢内 六角面一十五寸問外圓径幾何	As in the diagram, there is an outer circle (外) which and contains a hexagon. Say the hexagon sagitta is 24 *sun* and the sagitta inside the hexagon is 15 *sun*. Problem - what is the diameter of the outer circle?
答日　外圓径四十二寸	Answer: The diameter of the outer circle is 42 *sun*

[71]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

Modern Analysis

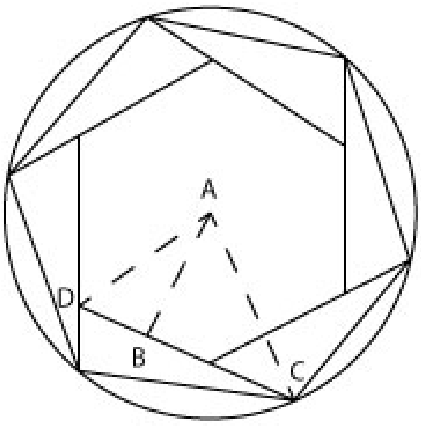


FIGURE 2.87: Suwa: Twentieth problem analysis

In Figure 2.86, the sagitta mentioned in the problem text are not labelled on the diagram. They are described in the text as the 'hexagon sagitta' and the 'sagitta inside the hexagon'. By treating the 'hexagon sagitta' as the lines which connect the vertices of the small and large hexagons - such a DC in Figure 2.85 - and the 'sagitta inside the hexagon' as the lines which connect the centre of the outer circle with the vertices of the small hexagon - such as AB - the answer on the tablet can be obtained.

In Figure 2.85, assume two right angle triangles $\triangle ABC$ and $\triangle ABD$. Let the diameter of the large circle be d. Finding the lengths of AB and BC allows AC to be obtained using the Pythagorean theorem. Since AC is equal to the radius of the large circle, doubling this gives the answer.

AB and BC can be found as follows

$$\begin{aligned}
&AD = 15 \\
&DC = 24 \\
&DB = \frac{AD}{2} \\
&AB = 15 \times \frac{\sqrt{3}}{2} \\
&BC = CB - DB = 16.5
\end{aligned} \tag{2.142}$$

From (2.142), applying the Pythagorean theorem to triangle $\triangle ABC$

$$
\begin{aligned}
&AC^2 = BC^2 + AB^2 \\
&AC^2 = 441 \\
&AC = 21 \\
&d = 42
\end{aligned}
\tag{2.143}
$$

Comments

A certain amount of contemplation and trial and error is required with this problem to determine the positions of the two sagitta since they are not explicitly labelled on the diagram and their description is not precise. This may suggest as with other problems an emphasis more on contemplation of the problem and its mathematical ideas than the solving of it.

2.8.14 Miharu Itsukushima Sangaku

FIGURE 2.88: The *sangaku* at Miharu Itsukushima shrine, Fukushima prefecture. (H. Kotera[72]).

The Miharu Itsukushima shrine 三春厳島神社 is located in the town of Miharu in Fukushima prefecture. It is of the *Itsukushima* vareity - such as the famous Miyajima Itsukushima shrine - which worship the sea deity *Susano*. This shrine contains six *sangaku* which form part of a coffered ceiling. Just one tablet is examined in this section, dedicated in 1885 by Watanabe Misao.

2.8.14.1 Sources and Transcription

The photograph used to transcribe this problem has been sourced from H. Kotera's *wasan* website. The original photograph used can be found at:

http://www.wasan.earth.linkclub.com/fukusima/miharuitukusima2.html.

[72]See http://www.wasan.earth.linkclub.com/fukusima/miharuitukusima2.html

A larger version of this image is located in D.11 in Appendix C. No previous transcriptions of this problem are known to be currently available.

2.8.14.2 Accompanying Text

After problem:

渡辺巳三郎 Watanabe Misao

2.8.14.3 Miharu Itsukushima Problem

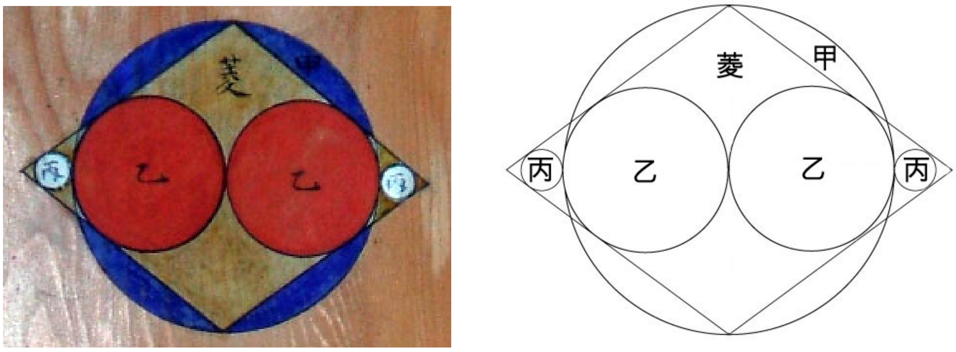


Figure 2.89: Left: Miharu Itsukushima problem. Right: Transcription. (H. Kotera[73]).

Translation

Transcription	Translation
今有如囨[74]甲圓與 菱重之丌[75]罅容乙 丙圓只云乙圓徑[76] 四寸問丙圓徑[77]如何	As in the diagram, there is a circle *kō* 甲 and a parallelogram containing circles *otsu* 乙 and *hei* 丙. Say that the diameter of circle *otsu* 乙 is 4 *sun*. Problem - what is the diameter of circle *hei* 丙?
畣[78]日丙圓徑一寸	Answer: The diameter of circle *hei* 丙 is 1 *sun*
術日置二分五厘乘乙圓徑 得丙徑合問	Technique: Put 2 *bu* 5 *rin* and multiply by means of the diameter of circle *otsu* 乙. Obtain the diameter of *hei* 丙 as required.

[73]See http://www.wasan.earth.linkclub.com/fukusima/miharuitukusima2.html
[74]This is an old variant of the character *zu* 圖 for figure/diagram.
[74]This is a variant of the character *sore* 其 which relates to the touching of figures.
[75]This is an older variant of *kei* 径 which refers to the diameter.
[76]Here the character for diameter has been used instead of the original character from the text. This is because the exact character in the text has been difficult to ascertain. The character can be seen to combine the radicals 又, 土, and 辶. However, such a character does not appear in modern dictionaries

Translation Notes

This *sangaku* text contains four very rare variants of characters which commonly appear on tablets. The decision of the author to use such archaic forms may suggest they were highly educated or interacted with a professional calligrapher with knowledge of old literary forms.

Technical Analysis

RENDERING 1 - PROCEDURE

Problem Say the diameter of circle *otsu* 乙 is 4 *sun*. What is the diameter of circle *hei* 丙?

Answer Circle *hei* 丙 $= 1$ *sun*

Solution 2 *bu* 5 *rin* $\times$ *otsu* 乙 $=$ *hei* 丙

hei 丙 $= 1$ *sun*

RENDERING 2 - FORMULA

Let the diameter the circle *otsu* 乙 $= a$, and the diameter of circle *hei* 丙 $= x$.

Problem Say $a = 4$. What is x?

Answer $x = 1$

Solution $x = 0.25a$

Modern Analysis

To find the value of circle *hei* 丙, a right angle triangle $\triangle ABC$ can be constructed as in Figure 2.90. Assume additional triangles $\triangle ADE$, $\triangle AFG$, and $\triangle AHI$. These are similar triangles, and since $\triangle ABC$ is proportional to $\triangle ADE$ the relation $\frac{BC}{DE} = \frac{DE}{FG}$ presents. Because two of the circles *otsu* 乙 make up the larger circle *kō* 甲, the diameter of *kō* 甲 is 8 and the length of BC is 4.

or lists of older variants. Variants on the character for diameter *kei* 径 include 経, 徑, and 逕. There is a kanji character 迳 (reading unknown) which visually appears similar and means to 'pass by/approach'. However given the context that the character appears in, which is after the characters for 'circle *otsu* 乙', I believe the author may have had a variation on the term for diameter in mind. For this reason I have placed the character *kei* 徑 used earlier in the text for diameter in this location.

[77]This is a variant of the character *toe* 富 which is synonomous with *kotae* 答 'answer'. See http://ksbookshelf.com/DW/Kanjirin/Kanjirin58.html for details.

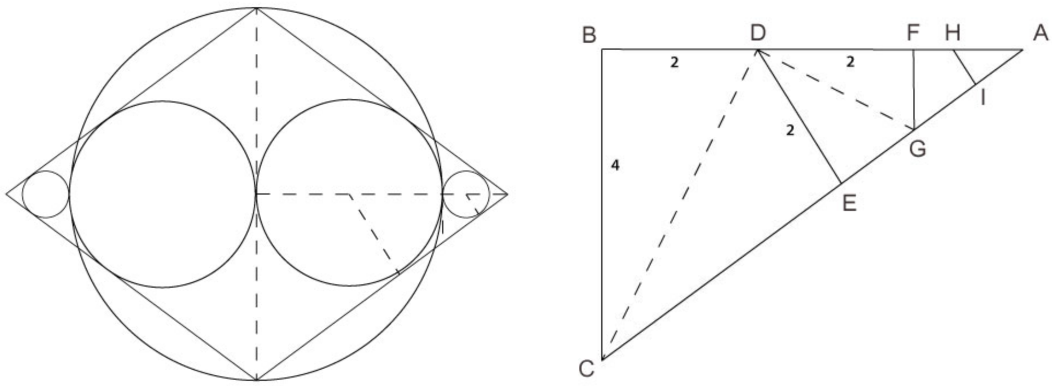


FIGURE 2.90: Miharu Itsukushima: Problem analysis

The lengths BD, EF, and ED, which are each half the diameter of *otsu* 乙, have a value of 2. The circle *hei* 丙 can then be determined by finding the length of HI, which is half *hei* 丙, such that

$$BC = 2DE$$

$$\frac{BC}{DE} = \frac{DE}{FG}$$

$$\frac{4}{2} = \frac{2}{DE}$$

$$\frac{DE}{FG} = \frac{FG}{H1}$$

$$\frac{2}{1} = \frac{1}{0.5}$$

$$2HI = 1$$

Chapter 3

Mathematical Functions of Sangaku

3.1 Introduction

The *sangaku* tradition saw people from a variety of classes in Japan develop and construct geometrical problems which they beautifully illustrated and placed in locations of religious and social significance. As shown in chapter 2, *sangaku* can be solved using Western mathematical methods. In both the Japanese and English language literature to date, it is these solutions employing Western methods which prevail. One such example is Fukagawa and Hidetoshi's popular *Sacred Mathematics: Japanese Temple Geometry*. Their presentation of *sangaku* and their solutions with Western mathematics prompted criticism by Peter J Lu, who desired to see original Japanese and Western methods side by side (see section 1.1) [47, p. 1050]. The techniques of the Japanese authors currently remains an open question. In response to this, I introduce some of the tools and methods of the *wasan* tradition in this chapter. I pay particular attention to the traditional Japanese *tenzan jutsu* method, which I apply to four *sangaku* to show how their problems can be solved with traditional Japanese techniques. Through this application it can be seen how the subject matter of *sangaku* matches other work found in *wasan* texts. I argue from this that *sangaku* problems should therefore be treated as general *wasan* problems. I also briefly discuss the claim made in section 1.1 by authors such as Horiuchi and Fukagawa that *sangaku* functioned as transmission devices. I support their view, but stress in chapter 4 that this is but one of their many functions.

3.2 Methods of the Wasan Tradition

As mentioned in section 1.4, *wasan* is the name given to the traditional Japanese mathematical activity of the Edo and early Meiji periods. In order to investigate how *sangaku* can be solved using original methods, I will briefly describe this tradition and the methods developed within it that may have been available to *sangaku* practitioners.

While being considered uniquely Japanese, *wasan* had its roots in the Chinese tradition. Illustratrating the use of Chinese mathematics by the Japanese, Hideyuki Majima has noted that Seki Takakazu - perhaps the most famous Japanese mathematician - transcribed and corrected the Chinese mathematician Yang Hui's 1275 *Yang Hui Suanfa* 楊輝算法 [49, p. 14]. Seki is also believed to have studied the Chinese *Suanfa Tongzong* by Cheng Dawei in 1592. It is also likely Seki had access to other Chinese mathematical texts which were based off works such as the *Nine Chapters on the Mathematical Art* 九章算經. Figure 3.1 shows a page from Liu Hui's third/fourth century commentary on the *Nine Chapters on the Mathematical Art* beside a page from the 1674 *Hatsubi Sanpo* 発微算法 by Seki.

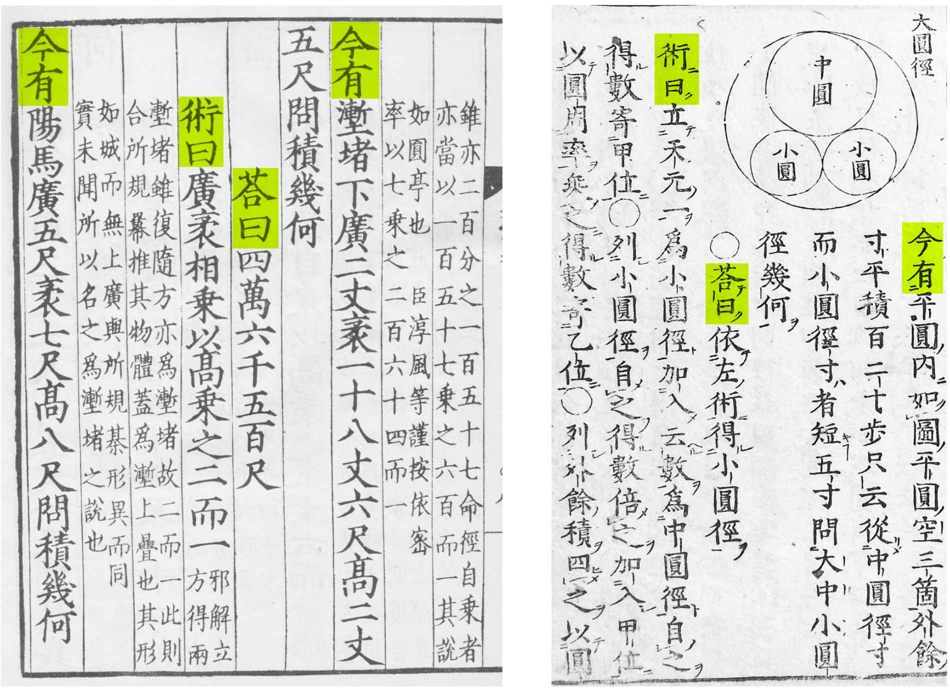


FIGURE 3.1: Left: *Nine Chapters on the Mathematical Art* commentary by Lui Hui. Right: Problem by Seki Takakazu in the *Hatsubi Sanpo*.

The main similarity between the work of Seki and Liu Hui is in terminology and presentation. Both texts give mathematical problems in three sections - problem, answer, technique. As highlighted in Figure 3.1, the problem sections begin with 今有, the answers with 答日, and the techniques with 術日. However, there some differences. One is the lack of drawn vertical lines in Seki's work, which can be seen in Figure 3.1. Also, he includes circular grammatical marks above the answer and in the technique section as a form of punctuation. A diagram pertaining to the problem is also given, and the answer section contains instructions to observe the technique section for the solution rather than giving a numerical answer. As well as this, Seki also solved some of his work differently using *bōsho hō* (later to become known as *tenzan jutsu*), which was a symbolic manipulation technique he had developed. Thus while greatly influenced by Chinese works in style and subject matter, in the work of Seki *wasan* can be seen as being something more than just Chinese mathematics.

Regarding the subject matter of *wasan*, it is known that the following topics were studied and known to Japanese mathematicians:

1. **Square and Cube Roots**
   Knowledge of square and cube roots in the *wasan* tradition date back to the early 1600s. Instructions on their calculation using the *soroban* are included in the 1627 *Jinkōki* of Yoshida Mitsuyoshi 吉田光由 (1598 - 1672).

2. **Calculation of Areas and Volumes**
   The Japanese knew how to calculate the areas of circles, hexagons, octagons and various other shapes. For instance, in the *Jinkōki* of Yoshida and *Sanpo Jojutsu* 算法助術 of Hiromu Hasegawa 長谷川弘 (1810-1887), coefficients and formulas pertaining to areas are presented. The *Jinkōki* in particular provides many examples of calculating the areas of various shaped fields. The volume of various shapes were also known in the tradition, with spheres being the most common subject of investigation.

3. **Pythagorean Theory**
   As mentioned in section 2.8.5.3, the Japanese knew of the Pythagorean theorem from the Chinese tradition. Lists of Pythagorean triples also occur in Japanese texts. For instance, the *Seiyo Sanpo* 精要算法 of 1781 contains a table of triples starting from 3-4-5 and ending with 372-925-997.

4. **Trigonometric Functions**
   As well as lists of Pythagorean triples, the Japanese also constructed tables relating to trigonometric functions. Chinese trigonometric tables were known in Japan under the title of *hassen* 八線, which means 'eight lines'. The 'eight lines' were the

eight functions of *sine*, *cosine*, *tangent*, *cotangent*, *secant*, *cosecant*, *versed sine*, and *coversed sine*. In Figure 3.2, these functions can be seen displayed in a diagram from the Japanese text 'Rules for using eight line diagram' *Hassenhyo Yohyo* 八線表用法 (date unknown).

Trigonometric functions most notably appear in the Japanese tradition in the text 'Calculation for using eight line diagram' *Hassenhyo Sanpo Kaigi* 八線表算法解義 by the well known mathematician Nakane Genkei 中根元圭 (1662-1733). Nakane Genki was a disciple of Takebe Katahiro and worked with him to translate and present 'Complete Treatise on Calender and Computation' to the *shōgun* in 1728 and 1733 [41, p. 361]. According to Kobayashi, the Tokugawa *shōgunate* ordered the same Chinese importer of this work to bring trigonometric function tables, for they had not been included in the text. Five books were sent, and soon after this "trigonometry and trigonometric function tables began to prevail among Japanese mathematicians and astronomers" [41, p. 363]. While the exact date of Nakane Genkei's publication of the tables is unknown, the introduction of this knowledge to Japan can be traced back to the 1730s and 1740s.

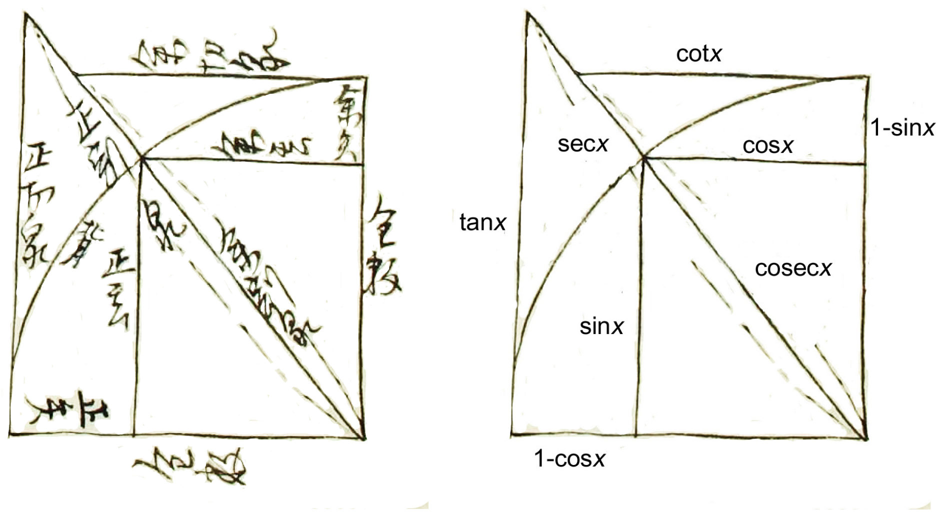


FIGURE 3.2: Left: Trigonometric functions in the *Hassen Yohyo* 八線表用法. Right: English translation.

5. **Rule of Cosines**

According to Fukagawa, a rule equivalent to the law of cosines appears often in the *wasan* tradition [29, p. 284]. Makishita explains that this law was known as *soukogen* [50, p. 147]. Makishita [50, p. 148] uses the following example to illustrate how *soukogen* functioned:

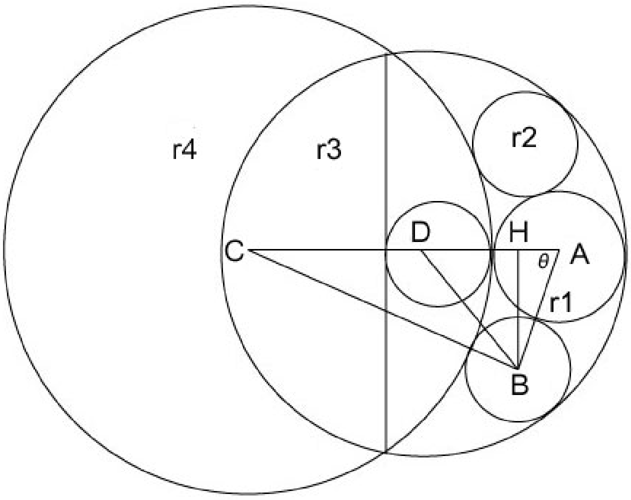


FIGURE 3.3: Applying the *soukogen*

Say in Figure 3.3 there are four different sized circles r_1, r_2, r_3, and r_4. From Figure 3.3

$$\begin{aligned}
AB &= r_1 + r_2 \\
AC &= r_1 + r_4 \\
BC &= r_2 + r_4 \\
AD &= r_3 - r_1 \\
BD &= r_3 - r_2 \\
AH &= y
\end{aligned} \tag{3.1}$$

From Figure 3.3, the cosine rule $c^2 = a^2 + b^2 - 2ab\cos\theta$ can be applied to $\triangle ABD$ to produce $BD^2 = AB^2 + AD^2 - 2AB \cdot AD \cos\angle BAD$. Inputting the values from (3.1) gives

$$(r_3 - r_2)^2 = (r_1 + r_2)^2 + (r_3 - r_1)^2 - 2(r_1 + r_2)(r_3 - r_1)\cos\angle BAD \tag{3.2}$$

Since $\angle BAD \equiv \angle BAH$, $\cos\angle BAD$ can be expressed in terms of the sides AH and BA of $\triangle ABH$

$$\cos\angle BAD = \frac{AH}{AB} = \frac{y}{(r_1 + r_2)} \tag{3.3}$$

Applying (3.3) to (3.2) produces the following expression

$$(r_1 + r_2)^2 + (r_3 - r_1)^2 - (r_3 - r_2)^2 - 2y(r_3 - r_1) = 0 \tag{3.4}$$

In a similar way, in $\triangle ABC$, the cosine rule can be applied to produce $BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cos \angle BAC$. As $\angle BAC \equiv \angle BAD \equiv \angle BAH$, formula (3.3) can be applied to produce

$$(r_1 + r_2)^2 + (r_1 + r_4)^2 - (r_2 + r_4)^2 - 2y(r_1 + r_4) = 0 \tag{3.5}$$

When combining $\triangle ABC$ and $\triangle ABD$, the y term from each equation can be eliminated. From this, an algebraic equation can be produced and used to solve for either r_1, r_2, r_3, or r_4. In this way the Japanese managed to calculate in a similar way to the law of cosines.

6. **Calculation of π**
   One topic which received much attention in the *wasan* tradition was the calculation of π. The earliest approximation to appear in a Japanese text was 3.16 in the 1627 edition of the *Jinkōki*. As the *wasan* tradition progressed Japanese practitioners began to improve upon this figure. For instance, Takebe Katahiro calculated π correctly to 42 places using a form of Taylor series expansion in the 1722 *Tetsujutsu Sankei*.

7. **Quadratic Equations**
   Using the *sangi* abacus mentioned in section 2.5, the Japanese were able to solve quadratic equations with one unknown. Sato explains that "equations were solved by placing about 6 cm long woodensticks, called counting rods, on a counting board made of paper" [81, p. 13]. Within later versions of the *Jinkōki* are also instructions on how to calculate the root x in the equation $x^2 + 20x = 19221$ on the *soroban*.

As well as these methods, Seki's *bōsho hō* was also available in the *wasan* mathematical tradition. Before examining how to solve some *sangaku* problems using traditional methods, the rules of *bōsho hō* will be explained.

3.2.1 Bōsho hō and Tenzan Jutsu

One of the major differences between the Chinese mathematical tradition and *wasan* was the development and use of *bōsho hō*. This technique, often referred to as the method of side-writing, was the symbolic manipulation technique Seki developed to solve equations with multiple unknowns. Seki built upon an existing counting board method known as *tengen jutsu* 天元術 which used the *sangi* 算木 rods mentioned in sections 2.5 and 3.2 to calculate an unknown. In their physical form, *sangi* counting rods were either coloured red to represent positive values or black for negative [53, p. 173].

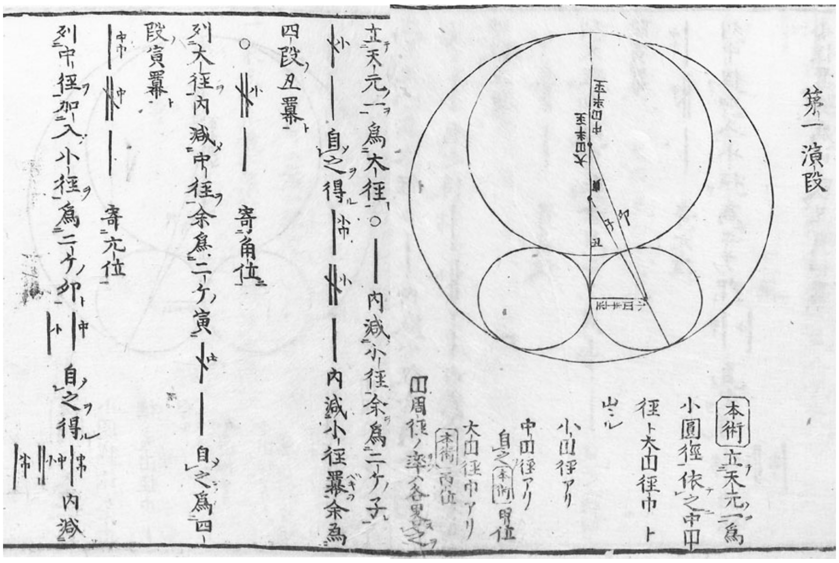


FIGURE 3.4: *Bōsho hō* in Seki's *Hatsubi Sanpo*

甲	a	$(+)\ a$
二 乙	$2\ b$	$(+)\ \frac{b}{2}$
$\cancel{\ }$小	$\cancel{\ }c$	- c

FIGURE 3.5: Left: Original *tenzan jutsu* calculation. Middle: Transcription. Right: Translation.

When presented in textbooks, negative numbers were presented by a line (which represented a rod) with a strike through. In some texts and pictures, they are represented by red lines, but due to printing difficulties woodblock print texts usually used the strike through system. In his work *Hatsubi Sanpo*, Seki introduced the *bōsho hō* system which used lines similar to the written form of the mentioned counting board method, but included kanji characters alongside to represent unknowns and express operations. The *bōsho hō* developed as the Edo period progressed and was later renamed *tenzan jutsu* 点竄術 in the work of the mathematician Matsunaga Yoshinsuke 松永良弼 (1693-1744), who was a second generation pupil of the Seki school [87, p. 104]. For the remainder of this chapter, I adopt Matsunaga Yoshinsuke's terminology - *tenzan jutsu* - to refer to this method of symbolic manipulation.

In *tenzan jutsu* negation was shown by a line being crossed through as in the written form of *tengen jutsu*. In general, the blank space between vertical or horizontal sets of lines indicated addition. To represent fractions, the numerator would be represented by the kanji character to the right of the line and the denominator by a character on the left. Figure 3.5 displays how *tenzan jutsu* expresses the terms a, $\frac{b}{2}$, and $-c$.

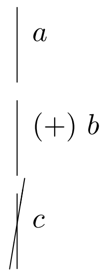


FIGURE 3.6: An example of vagueness in *tenzan jutsu*

There was an element of vagueness and ambiguity to *tenzan jutsu*. For example, the notation in Figure 3.6 can be interpreted as either expressing $a+b-c$, $a+b-c=0$, or ax^2+bx-c. Also, there is ambiguity caused by the different interpretations of expressions. For example, in the work of Aida Yasuaki and Ohara Toshiaki, there are instances in which calculations visibly appear as though they are presenting addition. However, for the calculations to make sense, they must actually be interpreted as alluding to division. For example, Figure 3.7 shows a set of calculations that appear in the Satimiya case study in section 3.3.3. The calculations appear to present $-(a \cdot b)+(a+b)+2(\sqrt{a}\sqrt{b})$. However when put in context, the correct interpretation should in fact be $\frac{-(a \cdot b)}{(a+b)+2(\sqrt{a}\sqrt{b})}$ (see section 3.3.3).

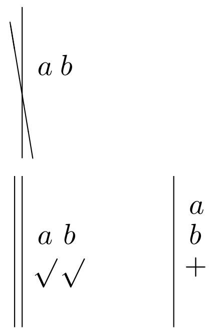


FIGURE 3.7: Example of division in *tenzan jutsu*

There are also instances where authors make mistakes in their calculations regarding positive and negative signs. Three such errors occur in problems examined in the case studies. In most these instances it appears the error occurs when forgetting to change the sign when carrying a term over from one side of the equation to another. These issues with vagueness and small mistakes mean that context and knowledge of the author is very important in the translation and intepretation of *tenzan jutsu* calculations. The observer must follow the calculations closely to see which interpretation is appropriate, as without context there may be several different interpretations.

Along with the characters described in section 2.8.5, other characters were used in *tenzan jutsu* whose roots trace back to Chinese cosmology. For instance, in *tenzan jutsu* texts auxiliary lines - as seen on the figure in Figure 3.4 - are often drawn on diagrams accompanying calculations, and these lines are labelled using characters associated with the twelve signs of the Chinese zodiac shown in Table 3.1.

No.	Character	Japanese	Sign
1	子	ne	Rat
2	丑	ushi	Ox
3	寅	tora	Tiger
4	兎/卯	u	Rabbit
5	辰	tatsu	Dragon
6	巳	mi	Snake
7	午	uma	Horse
8	未	hitsuji	Goat
9	申	sara	Monkey
10	酉	tori	Rooster
11	戌	inu	Dog
12	亥	i	Boar

Table 3.1: Signs of the Chinese Zodiac

Later in the 1769 text *Shuki Sanpo* 拾璣算法 by Arima Yoriyuki 有馬頼徸 (1714-1783), the equivalent of factorisation was added to the system. Arima's method involved giving terms to be factorised a specific label. These labels usually were derived from the *iroha*. The *iroha* - displayed in Table 3.2 - was an Heian period (794-1170 CE) poem. If terms were labelled with the same character from the *iroha*, this indicated the next step in the calculation was to combine them. For example, if terms such as ab and ac were labelled with i い, this would produce $a(b + c)$. More on this method can be found in section 3.2.2.

Japanese	Romaji
いろはにほへと	irohanihoheto
ちりぬるを	chirinuruwo
わかよたれそ	wakayotareso
つねならむ	tsunenaramu
うゐのおくやま	uwinookuyama
けふこえて	kefukoete
あさきゆめみし	asakiyumemishi
ゑひもせす	wehimosesu

Table 3.2: Iroha Poem

The *iroha* itself was a pangram similar to the English *The quick brown fox jumps over the lazy dog*, which contains every classical Japanese alphabet character exactly once[1]. The poem became associated with the "rudiments, basics (of something)" in the sense of the 'ABCs' in English [21, p. 169]. The *iroha* is sometimes used in modern Japan for numbering and labelling, and for the notes of an octave. This labelling system using the *iroha* in mathematics first appeared in the *Jinkōki*, which used the *iroha* to label rows on the *soroban*. It was also used by Seki in the *Hatsubi Sanpo* for labelling parts of an equation.

Along with the use of *iroha* for labelling elements, other symbols were sometimes used by later authors, indicating that *tenzan jutsu* was not standardised. For example, in the *Sanpo Shinsho* fourth edition of 1880, small triangles and circles are used in the place of *iroha*. However, at the core the methods are the same, and can be placed under the umbrella classification of *tenzan jutsu*. In the case studies of this chapter, only Ohara's work using *tenzan jutsu* will be used, as the tablets examined later in this section contain problems which appear in his work. However, before moving on to examine *sangaku* problems and *tenzan jutsu*, it is important to note the rules of calculation for *tenzan jutsu*, which are usually given as instructions to solve problems.

3.2.2 Rules of Tenzan Jutsu

To develop the following overview of the rules of calculation for *tenzan jutsu*, I have examined and translated the introduction sections of two 1810 works. The first is the *Sanpo Tenshoho Shinan* 算法天生法指南 by Aida Yasuaki 会田安明 (1747–1817), and the second the *Sanpo Tenzan Shinan* 算法点竄指南 by Ohara Toshiaki 大原利明 (? -1825). These instructions are a key part of *tenzan jutsu*, as they instruct the reader how to manipulate the *tenzan jutsu* symbols and lines to produce the desired outcome. They will be referred back to later when *tenzan jutsu* is applied to *sangaku* in the case studies.

3.2.2.1 自乗 Self Multiplication

The 'self multiplication rule' 自乗 is a means of squaring. For example, applying the rule to $2a$ produces $4a^2$. When there are two terms, squaring and expansion of the square occurs. For example, applying the rule to $a + b$ produces $a^2 + 2ab + b^2$. In the *Sanpo Tenzan Shinan* and *Sanpo Tenshoho Shinan*, examples of squaring two, three, four, and five terms are given.

[1] The characters *we* ゑ and *wi* ゐ are no longer used in the Japanese language.

3.2.2.2 括之 Put Together

The 'put together' rule 括之 is used to combine elements in various ways. Firstly it can be used to combine terms listed separately with different lines in the *tenzan* system to bring them under one line. For example, say we have the following addition

大	a	a
中	b	$(+)\ b$

When this rule is applied, the above juxtaposition is converted by addition into the single expression below

大 中 加	a b $+$	$a + b$

The rule is also used for combining like terms in the operation equivalent to factorisation discussed in section 3.2.1. In Ohara's work, the *iroha* system from section 3.2.1 is used to label where common terms appear. For example, in the *Sanpo Tenzan Shinan*, the sum

イ 大小	e $a\ c$	$a \cdot c$
イ 大中	e $a\ b$	$(+)\ a \cdot b$

Is converted by this rule to

イ	e	
小大 中 加	$b\ a$ c $+$	$a(b+c)$

It can be observed here that although brackets are included in my translation of this notation into Western algebra, the original notation of *tenzan jutsu* avoids any need for brackets.

3.2.2.3 解之 Splitting

The 'splitting' rule has the opposite function of the put together rule. It allows combined terms to be split into individual terms. For instance, if we have $a(b+c)$, the rule changes this into $(a \cdot b) + (a \cdot c)$.

Splitting is also the term used when substitution occurs. When a variable has been defined by a set of terms, the use of the splitting rule indicates that variable is to be decomposed into the terms that make it up. For example, say we have $y + c = x$ and know that $x = a + b$. The splitting rule indicates we can substitute the value of $a + b$ for x and change this expression into $y + c = a + b$.

3.2.2.4 遍省過乗 Eliminate Surplus Factors

When we have an equation such as $3x(a + b + c) - 6ab = 0$, where all terms have a common factor of 3, this rule indicates we are able to eliminate multiplication by three to produce $x(a + b + c) - 2ab = 0$.

3.2.2.5 同加異減 Add Same Subtract Different

This rule states that when we have two terms with similar signs, we can add them, but for different signs we subtract. For example, when there are two similar terms with positive signs - such as $+2ab + 2ab$ - they are combined to form $4ab$. This also applies for two negative values, such that $-2ab - 2ab$ become $-4ab$. However when there are similar terms which have different signs - such as $+a^2 - a^2$ - we subtract. For example, say we have $(a + b)^2 - (a - b)^2$ which produces $a^2 + 2ab + b^2 - a^2 + 2ab - b^2$. These can be grouped into similar terms to form $(+a^2 - a^2)$, $(+2ab + 2ab)$ and $(+b^2 - b^2)$.

3.2.2.6 變換 Conversion

The 'conversion' rule changes certain values into values that are equivalent. A common use is to convert a single number such as $-\sqrt{3}$ into $\sqrt{3}(\sqrt{3} - 2)(\sqrt{3} + 2)$. It indicates that a value is being replaced by a different expression of the same value.

3.2.2.7 乗除括之 Multiplication and Division Together and 加減括之 Addition and Subtraction Together

With the 'multiplication and division together' rule, when there is one fraction in an equation the rest of the equation is put over a common denominator. It converts an expression such as $\frac{a^2}{b} + 2a + b$ into $\frac{a^2}{b} + \frac{2ab}{b} + \frac{b^2}{b}$.

The rule of 'addition and subtraction together' allows for a negative term to be created in an equation with only positive terms. For instance, say we have $a^2 + ab + b^2$. This then has a negative term added by converting ab into $2ab - ab$ to give $a^2 + 2ab - ab + b^2$. This then becomes $(a + b)^2 - ab$. Another example of this is $a^2 + 2ab + b^2$, which can be converted into $a^2 + 4ab - 2ab + b^2$ and then $(a - b)^2 + 4ab$.

3.3 Sangaku Case Studies

In this section I show how the *tenzan jutsu* method can be applied to *sangaku* through a selection of case studies. In these case studies I use problems from the *Sanpo Tenzan Shinan* of Ohara to solve *sangaku* problems with *tenzan jutsu*. In the case studies, I first present an image of the tablet, then give the transcription and translation of the problem. Following this, the transcription and translation of similar problems from the *Sanpo Tenzan Shinan* are given to display the similarities between them.

I then present diagrams relating to the problems with auxiliary lines based on those found in the *Sanpo Tenzan Shinan*. I also provide a diagram which uses the variable names employed in my translation of the *tenzan jutsu* technique. Finally, the calculations from the *Sanpo Tenzan Shinan* are presented in three columns. The first is the transcription from the original text, the second a transliteration, and lastly a modern translation.

3.3.1 Case Study 1: Satimiya Sangaku

FIGURE 3.8: The *sangaku* at Satimiya shrine. (H. Kotera[2]).

The Satimiya *sangaku* was dedicated in 1824 to the Satimiya shrine, located in Takasaki city, Gunma prefecture. It was the collaborative work of many mathematical students, as the names of fifteen students - possibly from the local Yamada 山田 and Hideyoshi 英喜 schools - are listed on the tablet. The *sangaku* contains three problems, two of which - the middle and rightmost - are discussed in this case study. Versions of these two problems are also found in the *Sanpo Tenzan Shinan* where they are solved using *tenzan jutsu*. I first include the transcription and translation of each *sangaku* problem, followed by the solutions from the *Sanpo Tenzan Shinan* and a discussion on how the they can be applied to the *sangaku*.

3.3.2 Satimiya: First Problem

The first problem examined from the *Satimiya* tablet is located on the far right.

Translation

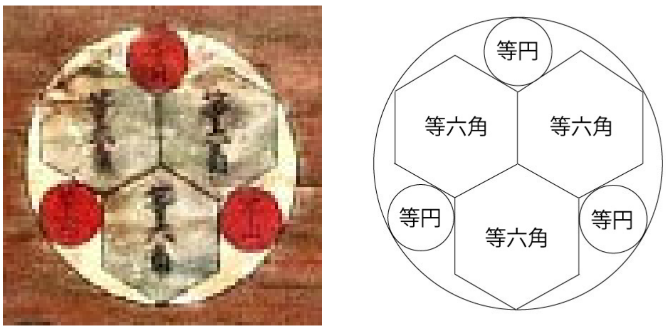


FIGURE 3.9: Left: First Satimiya problem. Right: Transcription. (H. Kotera[3]).

[2]See http://www.wasan.earth.linkclub.com/gunma/satimiya.html
[3]See http://www.wasan.earth.linkclub.com/gunma/satimiya.html

Transcription	Translation
今有如圖内等六角三個等圓三個只云外圓径一尺問等圓径幾何	As in the diagram, inside there are three equal sized hexagons and three equal sized circles (等). Say the diameter of the outer circle (外) is 1 *shaku*. Problem - what is the diameter of the equal sized circles?
答曰　等圓径二寸三分	Answer: The diameter of the equal sized circles is 2 *sun* 3 *bu*
術曰置三個開平方内減一個五分餘乘外径得等径合問	Technique: Put 3 *ko* and take the square root. Inside subtract 1 *ko* 5 *bu* and multiply the remainder by the diameter of the outer circle. Obtain the diameter of the equal circles as required.

3.3.2.1 Solving with Tenzan Jutsu

As mentioned, a version of this problem appears in the *Sanpo Tenzan Shinan* which is almost identical to that on the Satimiya tablet. The text from the *Sanpo Tenzan Shinan* for the problem reads:

Transcription	**Translation**
今有如圖圓内容等六角三個等圓三個只云外圓径一寸問等圓径幾何	As in the diagram, inside a circle are three equal sized hexagons and three equal sized circles (等). Say the diameter of the outer circle (外) is 1 *sun*. Problem - what is the diameter of the equal sized circles?
答曰　等圓径二分三厘	Answer: The diameter of the equal sized circles is 2 *bu* 3 *rin*
術曰置三個開平方内減一個五分餘乘外径得等径合問	Technique: Put 3 *ko* and take the square root. Inside subtract 1 *ko* 5 *bu* and multiply the remainder by the diameter of the outer circle. Obtain the diameter of the equal sized circles as required.

There is one distinct difference between the problem on the *sangaku* and that from the *Sanpo Tenzan Shinan*. While the Satimiya problem uses 1 *shaku* for the diameter of the outer circle, the *Sanpo Tenzan Shinan* gives this circle a diameter of 1 *sun*. However since 1 *shaku* equals 10 *sun*, the *Sanpo Tenzan Shinan's* solution with *tenzan jutsu* can still be applied to solve the *sangaku* problem because it uses a general formula.

The *Sanpo Tenzan Shinan* also differs by giving an additional diagram and solution using *tenzan jutsu*. This diagram - shown in Figure 3.10 - contains the original problem image with added auxillary lines. These lines are used to identify and label key lengths used in the calculation process. Following the diagram are the calculations themselves, which are transcribed and translated below.

Given Instructions:

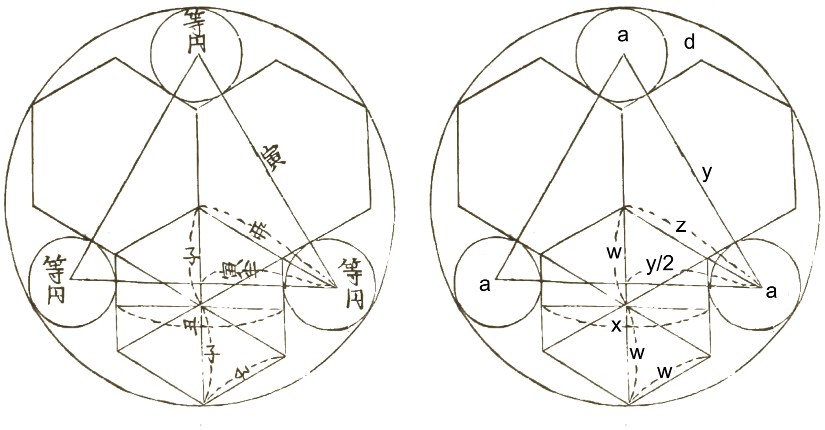


FIGURE 3.10: Left: Diagram from the *Sanpo Tenzan Shinan*. Right: Transcription.

[Let 等 be represented as a, 外 as d, 子 as w, 刃 as x, 寅 as y, and 兎 as z.]

解日置外四除

Solution: Put 外 [d] and divide it by 4.[4]

四	外	4	d	$\frac{d}{4} = w$
子		w		

置三個開平方乗子

Put 3 *ko* and take the square root. Multiply by 子 [w].[5]

四	三外 商	4	3 d $\surd$	$\frac{\sqrt{3}d}{4} = x$
刃		x		

[4]Here the large circle which contains the three hexagons and circles 等 [a] is divided into four. This is equivalent to the lenght 子 [w], which as displayed in Figure 3.10 is the side length of the hexagons.

[5]The width of the hexagons - labelled 刃 [x] in Figure 3.10 - is found through the formula $\sqrt{3} \times$ *side length* where the side length is the previously calculated 子 [w].

加等

Add [the diameter of] 等 [a].[6]

四	三外 商	4	3 d $\sqrt{}$	$\frac{\sqrt{3}d}{4}$
	等		a	$(+)\ a = \mathrm{y}$
寅		y		

以三個商除寅

Therefore divide 寅 [y] by means of 3 *ko* square root [$\sqrt{3}$].[7]

四	外	4	d	$\frac{d}{4}$
三 商	等	3 $\sqrt{}$	a	$(+)\ \frac{a}{\sqrt{3}} = z$
兎		z		

各三角之術也寄左 ◯ 又外半内減等半

For each triangle technique shift left. Futhermore, from half [the diameter of] 外 [d] subtract half [the diameter of] 等 [a].[8].

二	外	2	d	$\frac{d}{2}$

[6]In Figure 3.10, a triangle of side length 寅 [y] is drawn connecting the centers of the circles 等 [a]. By adding the previously calculated value 刃 [x] and the diameter of one circle 等 [a] the value of the triangle side length 寅 [y] is determined.

[7]The character 商 expresses a square root value, while the previously used characters 開平方 indicate the action of taking a square root. The value 兎 [z] on Figure 3.10 is found by dividing the previously determined 寅 [y] by $\sqrt{3}$.

[8]The symbol ◯ indicates the end of a paragraph, and has been translated as a full stop.

二	等	2	a	$-\frac{a}{2} = z$
兎		z		

與寄左相消

And gather on the left and cancel.[9]

四	外	4	d	$\frac{d}{4}$
三商	等	$\sqrt{3}$	a	$(+)\ \frac{a}{\sqrt{3}}$
二	外	2	d	$-\ \frac{d}{2}$
二	等	2	a	$(+)\ \frac{a}{2} = 0$
合矩		0		

遍乗除異減

Using multiply, divide, difference subtract [rule] many times.

	外三商		$d\ \sqrt{3}$	$-\ \sqrt{3}d$
𝍣	等	𝍣	a	$(+)\ 4a$

[9]Here the two expressions for z are placed on the left-hand side of the equation to eliminate z.

等三 　商	a 3 　 $\surd$	$(+)\ 2\sqrt{3}a = 0$
合矩	0	

括之

Put together.

外三 　商	d 3 　 $\surd$	- $\sqrt{3}d$
三等 商 加 二	3 a $\surd$ $+$ 2	$(+)\ 2a(\sqrt{3}+2) = 0$
合矩	0	

變換

Convert.

三三三外 商商商 加去 二二	3 3 3 d $\surd$ $\surd$ $\surd$ $+$ $-$ 2 2	$-d(\sqrt{3}((\sqrt{3}-2)(\sqrt{3}+2)))$
三等 商 加 二	3 a $\surd$ $+$ 2	$(+)\ 2a(\sqrt{3}+2) = 0$
合矩	0	

遍省過乘

Using elimination of higher power terms many times.[10]

三三外 商商 去 二	3 3 d $\surd$ $\surd$ $-$ 2	$-d(\sqrt{3}(\sqrt{3}-2))$

[10] In the previous step, we have $\sqrt{3}+2$ appearing twice. Twice it is multplied by other terms in the calculation. This step indicates that where we have multiple instances of a term like this we are to eliminate it. This is essentially cancelling out.

等	a	(+) $2a = 0$
合矩	0	

解之

Splitting.[11]

外三商	d 3 $\surd$	- $2\sqrt{3}d$
外	d	(+) $3d$
等	a	(+) $2a = 0$
合矩	0	

遍以二除之

Therefore using division by 2 many times.[12]

外三商	d 3 $\surd$	-$\sqrt{3}d$
一外 個 五 分	1 d *ko* 5 *bu*	(+) $d \times 1.5$

[11]Expanding of the brackets in $-d(\sqrt{3}(\sqrt{3}-2))$.

[12]This refers to the division by two across the equation that occurs. Division by 2 occurs for each term, which may be why we are told to do it many times.

等	a	$(+)\ a = 0$
定合矩	[Typically] 0	

如定例

As per the usual typical [technique].

一外 個 五 分	外三 商	1 d *ko* 5 *bu*	d 3 $\surd$

	1	$\sqrt{3}d - (d \times 1.5) = a$
等	a	

得等径式

Obtain 等 [a] formula.

Remarks

The method of the *Sanpo Tenzan Shinan* involves determining the value of the length z from the diagram in Figure 3.10. The value is able to be expressed in two different ways in terms of the diameters d and a of the outer and small inner circles. These expressions are $z = \frac{d}{4} + \frac{a}{\sqrt{3}}$ and $z = \frac{d}{2} - \frac{a}{2}$. Since these two expressions are equivalent they can be combined to form an algebraic equation from which the term a can then be solved for in terms of d.

The formula produced by the *Sanpo Tenzan Shinan* can be expressed in modern algebraic terms as follows

$$\sqrt{3}d - (d \times 1.5) = a \tag{3.6}$$

When inputting the values from the *sangaku* into this formula, the correct numerical answer can be derived. Therefore this solution using *tenzan jutsu* can be seen as a traditional Japanese solution to this *sangaku* problem.

One note however is that the *tenzan jutsu* calculation in the *Sanpo Tenzan Shinan* produced by applying the splitting rule to $-d(\sqrt{3}(\sqrt{3}-2))+2a=0$ appears to have incorrect signs. When expanded, this produces $2\sqrt{3}d-3d+2a=0$. However, in the *tenzan jutsu* calculation, this is given as $-2\sqrt{3}d+3d+2a=0$.

The difference here is the treatment of $2\sqrt{3}d$ as negative and $3d$ positive. The use of $2a-2\sqrt{3}d+3d=0$ gives $a=1.5d-\sqrt{3}d$, which does not match with the given instructions in the traditional technique section of the problem. Why there is this difference is unclear. It may be that negative signs were not treated rigidly by all authors. As stated in section 3.2.1, there can also be some ambiguity in the *tenzan jutsu* method. However given the author themselves gives a correct answer in written form in the technique section, they likely had $2a+2\sqrt{3}d-3d=0$ in mind.

This problem shows the connection in mathematical content between *wasan* treatises and *sangaku*. In fact, we see here that sometimes *sangaku* problems differ only in magnitude of values to other geometrical work done in the same period.

3.3.3 Satimiya: Second Problem

The second problem on the tablet is located in the middle of the three problems. It deals with three touching circles which in turn are touching the same line. This problem is found in both the texts of Aida and Ohara.

Translation

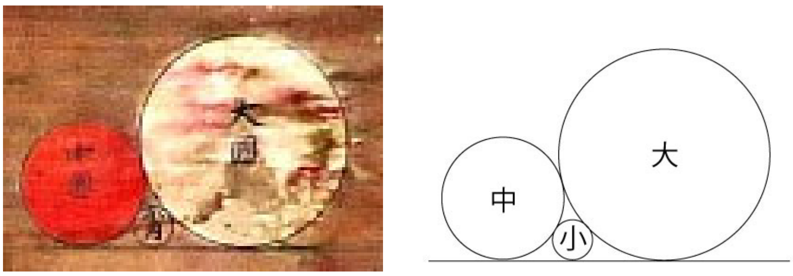


Figure 3.11: Left: Second Satimiya Problem. Right: Transcription. (H. Kotera[13]).

[13]See http://www.wasan.earth.linkclub.com/gunma/satimiya.html

TRANSCRIPTION	TRANSLATION
今有如図直線載大中二個其交罅容小円大径三十六寸中円径九寸小円径問幾何	As in the diagram, on a straight line there are two circles large (大) and medium (中) sized. Between them is contained a small (小) circle. The diameter of the large circle is 36 *sun*. The diameter of the medium circle is 9 *sun*. Problem - what is the diameter of the small circle?
答日　小円径四寸	Answer: The diameter of the small circle is 4 *sun*.
術日　置大円径乗中径名天開平方倍而加大中径和以除天得小径合問	Technique: Put the diameter of the large circle multiplied by the diameter of the medium circle. Name this heaven. Take the square root and double. Then add the diameters the large and medium circles. Divide into heaven. Obtain the diameter of the small circle as required.

3.3.3.1 Solving with Tenzan Jutsu

The same problem with slightly different wording appears in the *Sanpo Tenzan Shinan*. As with the previous problem, an additional diagram (Figure 3.12) and calcuations using *tenzan jutsu* are provided. This is how the problem in the text reads:

TRANSCRIPTION	TRANSLATION
今有如圖直線載大中二個其交罅容小圓大圓径三十六寸中圓径九寸問小圓径幾何	As in the diagram, on a straight line there are two circles large (大) and medium (中) sized. Between them is contained a small circle (小). The diameter of the large circle is 36 *sun*. The diameter of the medium circle is 9 *sun*. Problem - what is the diameter of the small circle?
答日　小圓径四寸	Answer: The diameter of the small circle is 4 *sun*.
術日　置大径乗中径名天方開平倍而加大中径和以除天得小径合問	Technique: Put the diameter of the large [circle] multiplied by the diameter of the medium [circle]. Name this heaven. Take the square root and double. Then add the diameters of the large and medium [circles]. Divide into heaven. Obtain the diameter of the small [circle] as required.

Given Instructions:

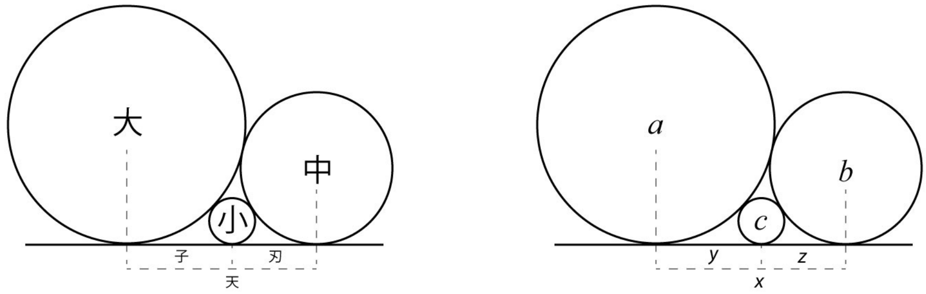


FIGURE 3.12: Left: Diagram from the *Sanpo Tenzan Shinan*. Right: Transcription.

[Let 大 be represented as a, 中 as b, and 小 as c.]

解日置一算命小圓径

Solution: Put one number and appointed diameter of 小 [c].[14]

| 小	| c	c

而依前術求各

Then using previous technique obtain each.[15]

| 大中 | 商商	| a b | $\sqrt{}\sqrt{}$	$\sqrt{a}\sqrt{b} = x$
天	*heaven*	

[14] This is an instruction to physically draw a line | of value 1 which will have the variable c associated with it. Essentially we are told to draw $|c$ - which just means $1c$ or c [37, p. 82].

[15] This particular problem uses a solution found in the problem prior to this one in the *Sanpo Tenzan Shinan* text. In the previous problem, just the circles 大 [a] and 中 [b] are present. This problem finds the length of the segment of the line between the two touching points, and the segment is labelled 天 [*heaven*]. The instructions here indicate that the technique used to find the length of the line *heaven* in the previous problem should be applied to find the lengths of the lines between the three circles 大 [a], 中 [b] and 小 [c]. Therefore the justification for the given values comes from a previous solution.

大小 商商	$a\ c$ $\surd\,\surd$	$\sqrt{a}\sqrt{c} = y$
子	y	
中小 商商	$b\ c$ $\surd\,\surd$	$\sqrt{b}\sqrt{c} = z$
刃	z	

而子刃和

Then add 子 [y] and 刃 [z].

大小 商商	$a\ c$ $\surd\,\surd$	$\sqrt{a}\sqrt{c}$
中小 商商	$b\ c$ $\surd\,\surd$	$(+)\sqrt{b}\sqrt{c} = x$
天	*heaven*	

寄左 ◯ 以天相消

Shift to the left. Therefore cancel 天 [*heaven*].[16]

大小 商商	$a\ c$ $\surd\,\surd$	$\sqrt{a}\sqrt{c}$
中小 商商	$b\ c$ $\surd\,\surd$	$(+)\sqrt{b}\sqrt{c}$

[16]The symbol ◯ appears in the original text and functions as a punctuation mark.

大中 商商	a b $\surd\,\surd$	$-\sqrt{a}\sqrt{b} = 0$
合矩	0	

左右分之

Split to left and right.[17]

大小 商商	a c $\surd\,\surd$	$\sqrt{a}\sqrt{c}$
中小 商商	b c $\surd\,\surd$	$(+)\sqrt{b}\sqrt{c}$
左爲	Put left	
大中 商商	a b $\surd\,\surd$	$-\sqrt{a}\sqrt{b}$
右爲	Put right	

左右自乘相消左冪與右冪適等也

Left and right squared and cancelled. Left squared and right squared. [There are] similar terms.[18]

イ	e	
大小	a c	$a \cdot c$

[17]The two sets of calculations that follow here are $\sqrt{a}\sqrt{c} + \sqrt{b}\sqrt{c}$ followed by the text 左爲 'put left' and $-\sqrt{a}\sqrt{b}$ followed by 右爲 'put right'. After this, they are squared and combined again. Given this, these instructions seem to indicate that certain terms are going to be temporarily seperated while an operation occurs. In this instance, it is squaring. When the squaring occurs, this seperation gives $(\sqrt{a}\sqrt{c} + \sqrt{b}\sqrt{c})^2 - (\sqrt{a}\sqrt{b})^2 = 0$ rather than $(\sqrt{a}\sqrt{c} + \sqrt{b}\sqrt{c} - \sqrt{a}\sqrt{b})^2 = 0$. The first calculation produces the result which follows later in the text of $a \cdot c + 2c(\sqrt{a}\sqrt{b}) + b \cdot c - a \cdot b = 0$.

[18]Here the procedure explained in the previous footnote occurs. The left and right-hand side are squared to produce $(\sqrt{a}\sqrt{c}+\sqrt{b}\sqrt{c})^2 = (-\sqrt{a}\sqrt{b})^2$. This is then expanded to produce $a \cdot c + 2c(\sqrt{a}\sqrt{b}) + b \cdot c - a \cdot b = 0$.

大中小 商商	$a\ b\ c$ $\surd\surd$	$(+)\ 2c(\sqrt{a}\sqrt{b})$
イ	e	
中小	$b\ c$	$(+)\ b \cdot c$
大中	$a\ b$	- $a \cdot b = 0$
合矩	0	

括之得

Put together and obtain.[19]

イ	e	
大小 中 加	$a\ c$ b $+$	$c(a+b)$
大中小 商商	$a\ b\ c$ $\surd\surd$	$(+)\ 2c(\sqrt{a}\sqrt{b})$
大中	$a\ b$	- $a \cdot b = 0$
合矩	0	

如例

As per usual [technique].[20]

[19]The verb *kukuru* 括る is sometimes used in modern Japanese to mean 'put in brackets', or 'factorise'. As the Edo Japanese did not use brackets in their work, I have translated this as 'put together'. The label イ [e] relates back to the method equivalent to factorisation discussed in section 3.2.1.

[20]This tells the reader that they next do what is a usual operation. In this instance, the calculation appears as though addition should occur - producing $-(a \cdot b) + (a+b) + 2(\sqrt{a}\sqrt{b})$. However as previously discussed in section 3.2.1, division occurs instead to produce $\frac{-(a \cdot b)}{(a+b)+2(\sqrt{a}\sqrt{b})}$.

大中		$a\ b$		- $a \cdot b$
大中 商商	大 中 加	$a\ b$ $\surd\,\surd$	a b $+$	$\div((a+b)+2(\sqrt{a}\sqrt{b})) = c$
得小径式		Obtain c formula		

絶答術則如左

Absolute answer technique [the answer is] shown left.

Remarks

The formula derived from the *Sanpo Tenzan Shinan* can be presented in modern terminology as

$$\frac{-\, a \cdot b}{((a+b)+2(\sqrt{a}\sqrt{b}))} = c \tag{3.7}$$

When inputting the values from the tablet - as well as the *tenzan jutsu* problem - into this formula a value of -4 rather than 4 is produced. This appears to be due to an error in the calculation, as either the formula should be $\frac{-ab}{(a+b)+2(\sqrt{a}\sqrt{b})} = -c$ or $\frac{ab}{(a+b)+2(\sqrt{a}\sqrt{b})} = c$. Given the authors both correctly describe the formula as $\frac{ab}{(a+b)+2(\sqrt{a}\sqrt{b})} = c$ in the technique sections of the text, it seems $-ab$ was written by mistake. When this last step is corrected, the technique does produce the correct solution and can be successfully applied to the *sangaku* to solve the problem using an original Japanese technique.

By examining the problem on the tablet and in the *Sanpo Tenzan Shinan*, it can be further seen how how *sangaku* problems can be solved using traditional *wasan* techniques. In this case, the instructions in the *Sanpo Tenzan Shinan* can be directly used to solve the Satimiya problem when the calculations are carried out correctly. Since the *Sanpo Tenzan Shinan* was published in 1810 - four years before the Satimiya tablet of 1824 - it may be the case that the *sangaku* author borrowed from this textbook. Nonetheless, it further shows how *sangaku* problems can be treated as general *wasan* problems not only in style but in content.

3.3.4 Case Study 2: Katayamahiko Sangaku

FIGURE 3.13: The *sangaku* at Katayamahiko shrine. (Image by author).

The Katayamahiko *sangaku* was dedicated in October of 1873 to the Katayamahiko shrine of Osafune, Okayama prefecture. It is an unusual tablet, with elaborate carving on the cedar wood, believed to have been done by a local carver. The tablet was created by the mathematician Irie Shinyuan, who was seventy-eight years old upon its creation [29, p. 145]. Hidetoshi and Fukagawa translate the preface to the tablet as follows:

> Mathematics is profound. People have their methods for solving problems. This is true in the West as well as in China and Japan. Those who do not study hard cannot solve any problems. I have not mastered mathematics yet, even though I have been studying from youth. And so I have not become a teacher for anyone, but some have asked me to teach mathematics to them. I showed them the solutions to the problems and will hang a sangaku at the Katayamahiko shrine nearby, on which sixteen problems are written. I dedicate this tablet to the shrine in the hope that my students may get more scholarship in mathematics [29, p. 145].

The problem examined in this section is the fourteenth problem (located sixth from the left on the second row) on the tablet.

Translation

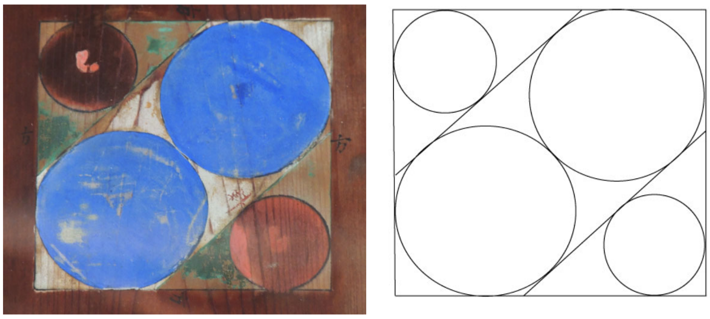


FIGURE 3.14: Left: Katayamahiko problem. Right: Transcription. (Image by author).

TRANSCRIPTION	TRANSLATION
今有如圖方内隔斜容甲乙円各二個只云甲円径一寸問乙円径幾何	As in the diagram, there is a square segmented by diagonal lines which contains two circles *kō* 甲 and *otsu* 乙. Say the diameter of circle *kō* 甲 is 1 *sun*. Problem - what is the diameter of circle *otsu* 乙?
答云　乙円径五分八厘五毛	Answer: The diameter of circle *otsu* 乙 is 5 *bu* 8 *rin* 5 *mo*
術日　置二個開平方以域二個余乘甲円径得乙円径合問	Technique: Put 2 *ko* and take the square root. Subtract 2 *ko*. Multiply the remainder by the diameter of circle *kō* 甲. Obtain the diameter of circle *otsu* 乙 as required.

3.3.4.1 Solving with Tenzan Jutsu

As with the Satimiya shrine *sangaku* problems, the *Sanpo Tenzan Shinan* can be consulted for a similar problem. The difference between the text of these two problems is the use of the character *en* 圓 for circle in the *Sanpo Tenzan Shinan*, and the more common variant *en* 円 used on the tablet. The text of the *Sanpo Tenzan Shinan* problem reads:

Transcription	Translation
今有如圖方内隔斜容甲乙圓各二個只云甲圓径一寸問乙圓径幾何	As in the diagram, there is a square segmented by diagonal lines which contains two circles *kō* 甲 and *otsu* 乙. Say the diameter of circle *kō* 甲 is 1 *sun*. Problem - what is the diameter of circle *otsu* 乙?
答云　乙圓径五分八厘五毛	Answer: The diameter of circle *otsu* 乙 is 5 *bu* 8 *rin* 5 *mo*
術日　置二個開平方以域二個余乘甲圓径得乙圓径合問	Technique: Put 2 *ko*. Take the square root. Subtract 2 *ko*. Multiply the remainder by the diameter of circle *kō* 甲. Obtain the diameter of circle *otsu* 乙 as required.

The *tenzan jutsu* calculations in the *Sanpo Tenzan Shinan* rely on finding different ways of calculating the length of the diagonal (labelled 方斜). The instructions are given below.

Given Instructions:

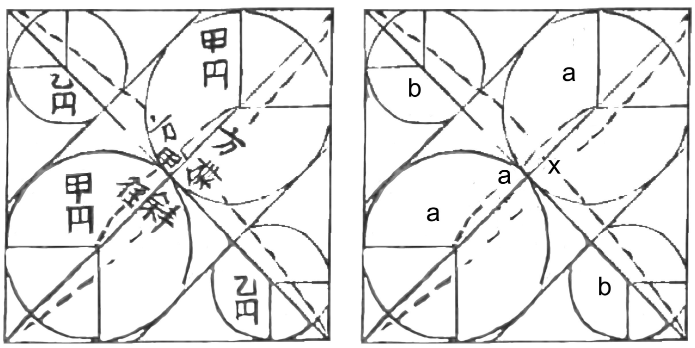


Figure 3.15: Left: Diagram from the *Sanpo Tenzan Shinan*. Right: Transcription.

[Let 甲 be represented as a, 乙 as b, and 方斜 (the diagonal) as x.]

解日置乙乘方斜率加甲乙和

Solution: Put 乙 [b] and multiply by the diagonal square ratio.[21]. Add 甲 [a] plus 乙 [b]

[21]The diagonal square ratio refers to $\sqrt{2}$.

乙二 商	b 2 $\surd$	$\sqrt{2} \cdot b$
甲 乙 和	a b $+$	
方斜	x	

寄左 ◯ 又置甲乘方斜率加甲

Move left. Furthermore, put 甲 [a] and multiply by the square diagonal ratio and add 甲 [a].

甲二 商	a 2 $\surd$	$\sqrt{2} \cdot a$
甲	a	$(+)\ a = x$
方斜	x	

與寄左相消求矩合

And move left and cancel to find equation.

乙二 商	b 2 $\surd$	$\sqrt{2} \cdot b$
甲 乙 和	a b $+$	$(+)\ a + b$
甲二 商	a 2 $\surd$	$-\sqrt{2} \cdot a$

甲 合矩	a 0	$-a = 0$

異域括之

Different terms put together.

二乙 商 加 一	2 b $\surd$ $+$ 1	$b(\sqrt{2}+1)$
二甲 商 合矩	2 a $\surd$ 0	$-a \cdot \sqrt{2} = 0$

變換

Convert.

二乙 商 加 一	2 b $\surd$ $+$ 1	$b(\sqrt{2}+1)$
二二二甲 商商商 加去 一一 合矩	2 2 2 a $\surd$ $\surd$ $\surd$ $+$ $-$ 1 1 0	$-a(\sqrt{2}((\sqrt{2}-1)(\sqrt{2}+1)))$

遍省過乘得

Using elimination of higher power terms many times obtain.

乙	b	b

二二甲 商商 去 一	2 2 a $\surd$ $\surd$ – 1	$-a(\sqrt{2}(\sqrt{2}-1)) = 0$
合矩	0	

解之

Splitting.[22]

乙	b	b
甲	a	- $2a$
二甲 商	2 a $\surd$	(+) $a\sqrt{2} = 0$
合矩定	0	

如定例求得乙径歸除式

As per the typical technique for obtaining the diameter of 乙 [b] using the division formula.[23]

二甲 商	甲	2 a $\surd$	a	$-2a + \sqrt{2}a = b$
得乙径式		Can find b formula		

故絶答術則如左

Therefore absolute answer technique [the answer is] shown left.

[22]We are to separate/expand the terms with a common factor. So $-a(\sqrt{2}(\sqrt{2}-1))$ becomes $-2a+a\sqrt{2}$.
[23]What this division formula refers to is not at this point entirely clear.

Remarks

The formula from the *Sanpo Tenzan Shinan* gives in modern terminology

$$-2a + \sqrt{2}a = b \tag{3.8}$$

This should in fact read $-2a + \sqrt{2}a = -b$, but the author has made an error in the second to last step of the *tenzan jutsu* calculations and has not converted b to $-b$ when shifting the term to the right-hand side of the equation. From $-2a + \sqrt{2}a = -b$ the value of $2a - \sqrt{2}a = b$ as found in the technique sections of the *sangaku* and *Sanpo Tenzan Shinan* can be derived. When entering the numerical values, this gives 0.585786.

The result of $2a - \sqrt{2}a = b$ is also found in Fukagawa and Rothman's *Sacred Mathematics: Japanese Temple Geometry*. However, Fukagawa and Rothman only show the solution to the problem in terms of modern mathematics. By examining texts of the period like the *Sanpo Tenzan Shinan*, a original traditional method for finding the solution can be found. This helps to better understand Fukagawa and Rothman's result, and also how the problem could have been originally done. This problem also evidences how *sangaku* problems can be connected back to the broader *wasan* tradition through their mathematical content.

3.3.5 Case Study 3: Mansyouin Sangaku

FIGURE 3.16: The *sangaku* at Mansyoin temple. (Image by author).

The Mansyouin *sangaku* was dedicated in 1852 and is located in the village of Kijimadaira in the Shimotakai district of Nagano Prefecture.

Versions of the two problems featured on this tablet are found side by side in the *Sanpo Tenzan Shinan*. As with the previously examined case studies, the *Sanpo Tenzan Shinan* presents an additional image with auxiliary lines and *tenzan jutsu* calculations.

3.3.6 Mansyouin: First Problem

The first problem examined is that on the right-hand side of the tablet. It deals with four touching circles. In the middle of these circles lies a fifth, and it is this circle's diameter that is sought in the problem.

Translation

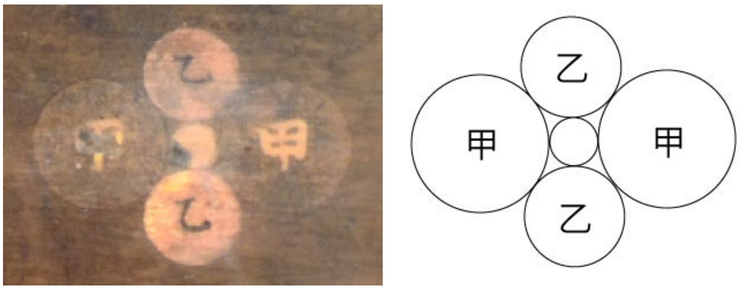


FIGURE 3.17: Left: First Mansyouin Problem. Right: Transcription. (Image by author).

TRANSCRIPTION	TRANSLATION
今有如圖以甲乙圓各二個丙圓徑圍只云甲圓若干乙圓若干問丙円幾何	As in the diagram, there are two circles each of *kō* 甲 and *otsu* 乙 which enclose a circle *hei* 丙. Say that the value of circle *kō* 甲 is known. Say the value of circle *otsu* 乙 is known. Problem - what is the value of circle *hei* 丙?
答日　丙圓徑一寸	Answer: The diameter of the circle *hei* 丙 is 1 *sun*
術日　置甲圓乘乙圓名子列甲圓加乙圓徑半之名丑自之加子開平方以商域丑得丙圓徑合問	Technique: Put *kō* 甲 and multiply by *otsu* 乙. Name this *ne* 子.[24]Add *kō* 甲 and *otsu* 乙 and halve. Name this *ushi* 丑. Self multiply and add *ne* 子. Take the square root. Subtract *ushi* 丑. Obtain the diameter of circle *hei* 丙 as required.

.

3.3.6.1 Solving with Tenzan Jutsu

As mentioned, a similar problem to the *sangaku* is found in the *Sanpo Tenzan Shinan*. In the *Sanpo Tenzan Shinan*, the value of the circle *kō* 甲 is sought in terms of the circles *otsu* 乙 and *hei* 丙. This differs to the *sangaku*, where the circle *kō* 甲 is sought in terms of the circles *otsu* 乙 and *hei* 丙. The *Sanpo Tenzan Shinan* also gives numerical values for the circles *otsu* 乙 and *hei* 丙, while the tablet only gives the numerical value of the circle *hei* 丙 in the answer section. As will be discussed later, the formula given in the technique section of the *sangaku* can only work when we have numerical values for the circles *otsu* 乙 and *hei* 丙.

[24]The character 子 is commonly read as *ko* and translated as 'child'. However the character 丑 also appears, and when in series these two characters express the first two animals of the Chinese zodiac rat *ne* 子 and ox *ushi* 丑. Due to this the character 子 has been read *ne* instead of *ko*.

Transcription	Translation
今有如圖以甲乙圓各二個圍丙圓只云乙圓徑二寸丙圓徑寸一問甲圓徑幾何	As in the diagram, there are two circles each of *kō* 甲 and *otsu* 乙 which enclose a circle *hei* 丙. Say that the value of circle *otsu* 乙 is 2 *sun*. Say the value of circle *hei* 丙 is 1 *sun*. Problem - what is the value of circle *kō* 甲?
答日　甲圓徑三寸	Answer:　The diameter of circle *kō* 甲 is 3 *sun*
術日　置以乙丙徑差除乙丙徑加乘丙徑得甲徑合問	Technique: By means of *otsu* 乙 minus *hei* 丙, divide into *otsu* 乙 plus *hei* 丙. Multiply by *hei* 丙. Obtain the diameter of circle *kō* 甲 as required.

Given Instructions:

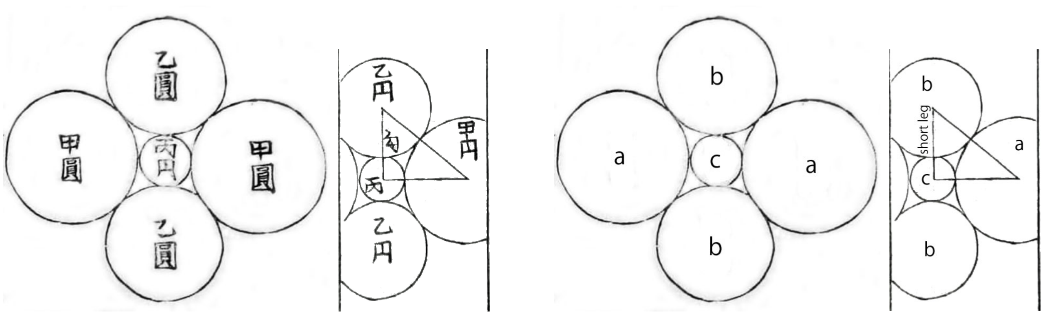


Figure 3.18: Left: Diagram from the *Sanpo Tenzan Shinan*. Right: Transcription.

[Let 甲 be represented as a, 乙 as b, and 丙 as c.]

解日置一算命甲径

Solution. Put one number and appointed diameter of 甲 [a].

	甲		a	a

依圖求各

From picture find all.[25]

二	乙	2	b	$\frac{b}{2}$

[25]This refers to the triangle in Figure 3.18.

二	丙	2	c	$(+)\ \frac{c}{2} =$ short leg
勾		*short leg*		

二	甲	2	a	$\frac{a}{2}$
二	丙	2	c	$(+)\ \frac{c}{2} =$ long leg
股		*long leg*		

二	甲	2	a	$\frac{a}{2}$
二	乙	2	c	$(+)\ \frac{b}{2} =$ hypotenuse
弦		*hypotenuse*		

而勾冪股冪和寄左 ◯ 以弦冪相消同加異減得

Then square 勾 [short leg] and square 股 [long leg], add, and shift left. Therefore 弦 [hypotenuse] cancel and add same, subtract different and obtain.[26]

四	丙乙	4	$b\ \ c$	$\frac{2bc}{4}$
四	丙 巾	4	c 2	$(+)\ \frac{2c^2}{4}$

[26]Here the Pythagorean theorem is applied to produce *short leg*^{2} + *long leg*^{2} = *hypotenuse*^{2}. Then the hypotenuse squared is moved to the left, and expansion, cancellation, and rearranging occurs.

四 ‖ 甲丙	4 ‖ a c	$(+)\ \dfrac{2ac}{4}$
四 ‖ 乙甲	4 ‖ a b	$-\ \dfrac{2ab}{4} = 0$

遍省過乗括之

Using elimination of higher power terms many times, put together.[27]

乙丙 丙 加	b c c $+$	$c(b+c)$
乙甲 丙 差	b a c $-$	$-\ a(b-c) = 0$

如例

As per usual [technique].

乙丙 丙 加	b c c $+$	$c(b+c)$
乙 丙 差	b c $-$	$\div b - c =$ a

得甲径式 Obtain a formula

[27]Here the fraction is eliminated, and the multiplication by two occurring on each of the terms is also cancelled out. Factorisation also occurs.

Remarks

The *Sanpo Tenzan Shinan* gives a formula which literally appears to read $c(b+c)-(b-c)=a$. However this formula cannot be produced from the previously given calculation of $c(b+c)=a(b-c)$. The a term however can be separated to the right-hand side by carrying by over $(b-c)$ from the right to the left-hand side. This creates an operation of division which produces $\frac{c(b+c)}{b-c}=a$. This division fits with what is written in the technique section of the problem, which reads "By means of *otsu* 乙 minus *hei* 丙, divide into *otsu* 乙 plus *hei* 丙. Multiply by *hei* 丙". Because of this, I believe the author intended for a division operation to occur here.

Interpreting the final *tenzan jutsu* calculation as involving an operation of division, the formula can be expressed as

$$\frac{c(b+c)}{b-c}=a \tag{3.9}$$

In the Mansyouin *sangaku* problem text, the technique given for finding the solution in modern notation differs. This is due to the fact that the term c is sought instead of the term a. It reads

$$\sqrt{\left(\frac{a+b}{2}\right)^2+ab}-\left(\frac{a+b}{2}\right) \tag{3.10}$$

While formula (3.10) is provided in the technique section of the *sangaku*, the problem is underdetermined. This is because the numerical answer given in the answer section (1 *sun*) cannot actually be obtained with this formula. This is because no numerical information is given for the values a and b. This is not an issue with the problem from the *Sanpo Tenzan Shinan*, which includes all the necessary values in the text.

The authors of the 2005 booklet *Sangaku of Kijimadaira* 木島平村の和算 believe that the finding of a minimum value for c when a, b, and c are natural numbers could be the real intention [89, p. 34]. This would mean that a would equal 3 *sun* and b would equal 2 *sun*. When entering these values into the formula above, the value of 1 *sun* is indeed produced. Since the values from the Mansyouin problem are the same as the *Sanpo Tenzan Shinan*, and both problems on the *sangaku* are side by side in the in this mathematical treatise, the author may have studied the problems of the *Sanpo Tenzan Shinan* and derived their problems directly from it. This may show a further connection between *wasan* texts and *sangaku*, and the knowledge *sangaku* authors may have had of *tenzan jutsu*.

The traditional method from the *Sanpo Tenzan Shinan* can also be rearranged and applied to solve the Mansyoin problem. To do this, the first four steps from the *Sanpo Tenzan Shinan* are followed. The equation is then rearranged to solve for c as follows

$$\begin{aligned}
&\frac{2bc}{4}+\frac{2c^2}{4}+\frac{2ac}{4}-\frac{2ab}{4}=0\\
&c^2+c(a+b)=ab\\
&c^2+c(a+b)+\left(\frac{a+b}{2}\right)^2=ab+\left(\frac{a+b}{2}\right)^2\\
&\left(c+\frac{a+b}{2}\right)^2=ab+\left(\frac{a+b}{2}\right)^2\\
&c+\frac{a+b}{2}=\sqrt{ab+\left(\frac{a+b}{2}\right)^2}\\
&c=\sqrt{\left(\frac{a+b}{2}\right)^2+ab}-\left(\frac{a+b}{2}\right)
\end{aligned}\tag{3.11}$$

3.3.7 Mansyouin: Second Problem

Translation

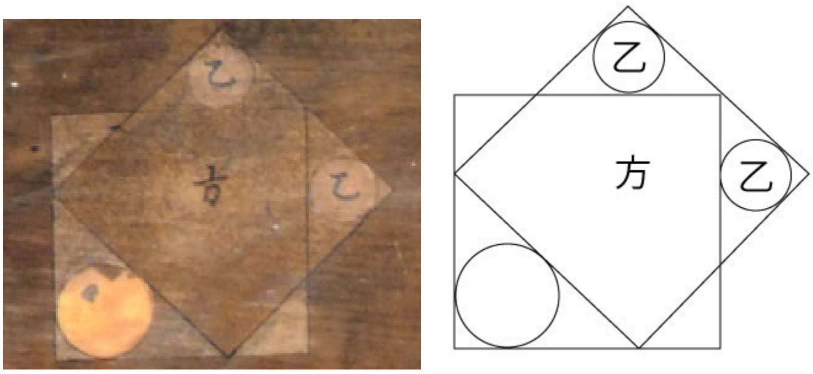


FIGURE 3.19: Left: Second Mansyouin Problem. Right: Transcription. (Image by author).

Transcription	Translation
今有如圖大小方内容甲圓一個乙圓二個只云乙圓若子干問甲幾何	As in the diagram, there are large and small squares which contain one circle *kō* 甲 and two circles *otsu* 乙. Say that the value of the circles *otsu* 乙 is known. Problem - what is [the diameter of circle] *kō* 甲?
答日甲圓径二寸一分	Answer: The diameter of circle *kō* 甲 is 2 *sun* 1 *bu*
術日　置乙圓径自之信之以開平方得甲円径合問	Technique: Put the diameter of circle *otsu* 乙 and self multiply. Double and take the square root. Obtain the diameter of circle *kō* 甲 as required.

3.3.7.1 Solving with Tenzan Jutsu

As with the first problem on the Mansyouin *sangaku*, this problem also appears underdetermined. This is because a numerical value is given in the answer section but the value (in this case *kō* 甲) required to obtain that answer is not given. This time the *Sanpo Tenzan Shinan* also appears underdetermined. It once again differs in the figure it seeks, which this time is the value of the circle *otsu* 乙.

Transcription	Translation
今有如圖大小方交内容甲圓及乙圓二個甲圓径一寸問乙圓径幾何	As in the diagram, there are intersecting large and small squares which contain two circles *kō* 甲 and *otsu* 乙. The diameter of circle *kō* 甲 is 1 *sun*. Problem - what is the diameter of circle *otsu* 乙?
答日乙圓径七分 ◯ 七一有奇	Answer: The diameter of circle *otsu* 乙 is 7 *bu* 071...
術日　置五分開平方乗甲径得乙径合問	Technique: Put 5 *bu* and take the square root. Multiply by the diameter of circle *kō* 甲. Obtain the diameter of *otsu* 乙 as required.

Given Instructions:

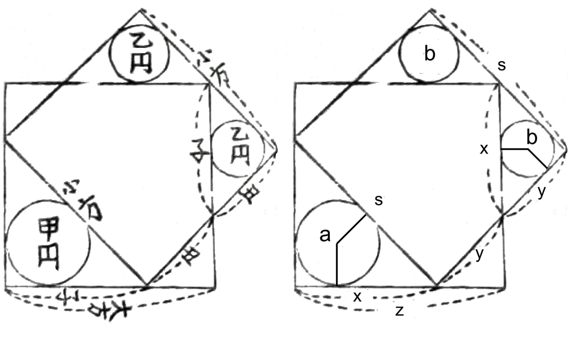


Figure 3.20: Left: Diagram from the *Sanpo Tenzan Shinan*. Right: Transcription.

[Let 甲 be represented as a, 乙 as b, 小方 as s, 子 as x, and 刃 as y]

解日置一算命二位

Solution put one number and appointed two (standing values).

小方	乙	*small side*	b	$s(+)b$

而依勾股内容圓術求子刃

Then using the circles contained inside 勾 [short leg] and 股 [long leg] find technique for 子 [x] and 刃 [y].[28]

二	甲	二	小方	2	a	2	*small side*	$\frac{s}{2}(+)\frac{a}{2}$ = x
	子				x			
二	乙	二	子	2	b	2	x	$\frac{x}{2}(+)\frac{b}{2}$ = y
	刃				y			

倍之

Double.[29]

乙	子	b	x	$b(+)x$ [= s]

[28]From the auxiliary lines drawn on two of the circles in Figure 3.20, it can be seen that half the diameter of 甲 [a] and half the length 小方 [s] make up the line 子 [x]. Also half of the circle 乙 [b] and half the length 子 [x] form the length 刃 [y].

[29]Here the line 刃 [y] is doubled. This is equal to 小方 [s], although it is not directly stated in the text.

小方寄左 ◯ 以小方相消

Small side move left. By means of 小方 [s] cancel.[30]

子	x	x
乙	b	(+) b
小方	*small side*	- $s = 0$
合矩	0	

解子

Splitting 子 $\left[\frac{s}{2} + \frac{a}{2}\right]$.[31]

二 小方	2 *small side*	$\frac{s}{2}$
乙	b	(+) b
二 甲	2 a	(+) $\frac{a}{2}$
小方	*small side*	- $s = 0$
合矩	0	

[30] Since $x + b = 2y$, and $2y = s$, it is determined that $x + b = s$. Both sides then have s subtracted.

[31] Splitting rule is used in this instance to 'split' the value of 子 [x] into its parts $\frac{s}{2} + \frac{a}{2}$, which are substituted in.

遍乘除異減

Using multiply, divide, difference, subtract many times.[32]

乙	b	$2b$
甲	a	$(+)\ a$
小 方	*small* *side*	- $s = 0$
合矩	0	

括之

Put together.

乙 二 甲 加	b 2 a $+$	$2b + a$
小 方	*small* *side*	- $s = 0$
合矩	0	

故求

Therefore find.

乙 二 甲 加	b 2 a $+$	$2b + a = s$
方小	small side	

[32]Multiplying by 2 to get rid of the fractions, and so on.

而依圖求

Then using diagram find.[33]

子	小方	x	*small side*	$\frac{s}{a} = \frac{x}{b}$
乙	甲	b	a	
矩同		Same equation		

斜乘相消求矩合

Multiply line, cancel, and find equation.[34]

小乙方	*small side* b	- $(b \cdot s)$
子甲	x a	$(+)$ $x \cdot a = 0$
合矩	0	

解子

Splitting 子.[35]

二	小甲方	2	*small side* a	$\frac{a \cdot s}{2}$
二	甲巾	2	$\frac{a}{2}$	$(+)$ $\frac{a^2}{2}$

[33] Here it is determined from Figure 3.20 that the ratio of the circle 甲 [a] and line 小方 [s], along with the circle 乙 [b] and line 子 [x] are equal such that $s : a = x : b$ or $\frac{s}{a} = \frac{x}{b}$.

[34] Using multiplication, $\frac{s}{a} = \frac{x}{b}$ can be expressed as $xa = bs$. After this has been found, bs is then subtracted from both sides to produce $xa - bs = 0$.

[35] Substitute $\frac{s}{2} + \frac{a}{2}$ for 子 [x].

小乙 方	*small b* *side*	- $(b \cdot s) = 0$
合矩	0	

遍乘除象

Using multiplication and division many times.

小甲 方	*small a* *side*	$a \cdot s$
甲 巾	a 2	$(+)\ a^2$
小乙 方	*small b* *side*	- $2(b \cdot s) = 0$
合矩	0	

解小方

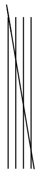
Splitting 小方 [s].[36]

乙甲 二 甲 加	$b\ \ a$ 2 a $+$	$a(2b + a)$
甲 巾	a 2	$(+)\ a^2$
乙乙 二 甲 加	$b\ \ b$ 2 a $+$	- $2b(2b + a)$
合矩	0	

[36]Replace 小方 [s] for $2b + a$.

解和同加異減

Splitting sum, add same subtract different.

甲巾	$\underset{2}{a}$	$2a^2$
 乙巾	 $\underset{2}{b}$	- $4b^2 = 0$
合矩	0	

遍省二

Using elimination of two many times.[37]

甲巾	$\underset{2}{a}$	a^2
 乙巾	 $\underset{2}{b}$	- $2b^2 = 0$
合矩	0	

故

Therefore.

二 甲巾	2 $\underset{2}{a}$	$\frac{a^2}{2} = b^2$
乙径冪	b^2	

[37]Divide by two and then shift b terms to the right-hand side of the equation. Divide by two again.

開平方

Take the square root.

二商	甲	2 √	a	$\frac{a}{\sqrt{2}} = b$
乙径		b		

Remarks

As noted earlier, the *sangaku* problem produces a numerical answer even though no values are given in the problem for use in the formula. In this instance, the *Sanpo Tenzan Shinan* also does not give the necessary values for obtaining a numerical value as an answer. However, working backwards from the answer in the *Sanpo Tenzan Shinan* problem gives $7.071 \times \sqrt{2}$. This produces a numerical value that can be approximated as 10. Working backwards from the *sangaku* gives 1.48492424.

Regardless of numerical values, the formula from the *Sanpo Tenzan Shinan* can be rearranged and applied to produce a formula for finding the solution to the problem on the Mansyouin *sangaku* as such

$$\frac{a}{\sqrt{2}} = b$$

$$b \cdot \sqrt{2} = a$$

This shows that both problems on this *sangaku* are solvable using traditional techniques of *wasan*. Also, it evidences how geometrical principles studied in *wasan* texts such as the *Sanpo Tenzan Shinan* were also researched by *sangaku* authors. In this instance, the author shows a deep understanding of the geometry behind the *Sanpo Tenzan Shinan* problem, for they alter the problem and obtain the value of a different figure to that given in the text. This shows the sophistication of some *sangaku* authors, as their problems were on par with work appearing in textbooks and they had the ability to alter and reinterpret *wasan* problems.

3.3.8 Case Study 4: Nagano Tenman-gū Sangaku

FIGURE 3.21: The *sangaku* at Nagano Tenman-gū shrine. (Image by author).

The Nagano Tenman-gū *sangaku* is located in the village of Kijimadaira in the Shimotakai district of Nagano Prefecture. It was dedicated to a small wooden shrine enshrining the Japanese god *Tenjin*, a deity of education and science. The Nagano Tenman-gū shrine is rests on a wooded hillside on the outskirts of Kijimadaira, whose population the council website put at 5,312 in 2005. It is located roughly 262 km from Tokyo, and 40 km from Nagano city.

The shrine contains three *sangaku* mathematical tablets, all believed to have been dedicated in 1888 [89, p. 5]. This dedication date is only given on one of the three tablets, but their similar size, condition, construction, and installation by a professional carpenter has led members of the Kijimadaira community to believe all were all dedicated at the same time. From the three tablets in this shrine, I have selected one, dedicated by Kobayashi Hirokichi 小林廣吉 (dates unknown), to examine as a case study. This is because Kobayashi Hirokichi's tablet contains problems similar to the *Sanpo Tenzan Shinan*.

3.3.9 Nagano Tenman-gū: First Problem

The first problem of the tablet of Kobayashi Hirokichi relates to a series of circles tangent to or inside one another.

Translation

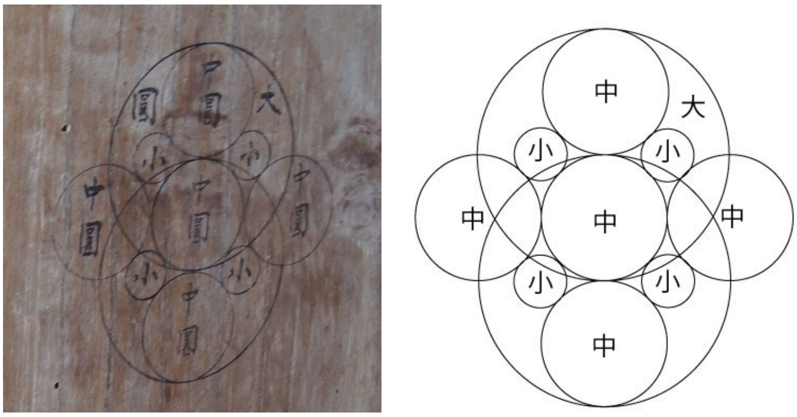


FIGURE 3.22: Left: First Nagano Tenman-gū Problem. Right: Transcription. (Image by author).

TRANSCRIPTION	TRANSLATION
今如圖大圓二個中圓五個交罅小圓容有只云大圓壹尺問小圓径幾何	As in the diagram, there are two large circles (大) and five medium circles (中). In the space between where these circles intersect are small circles (小). Say the diameter of the large circles is 1 *shaku*. Problem - what is the diameter of the small circles?
答曰　小圓二寸 0 七分有奇	Answer: The diameter of the small circles is 2 *sun* 0 7 *bu* ...
術曰　置五個開平方内減五個餘乗大圓径小径得合問	Technique: Put 5 *ko* and take the square root. Inside subtract 5 *ko*. Then multiply by the diameter of the large circles. Obtain the diameter of the small circles as required.

To note, the given value of 2 *sun* 07 *bu* for the answer is impossible given the Japanese number system described in section 2.6. Here it will be remembered that the units used by *sangaku* authors are *shaku* 尺 (1 *shaku* = 10 *sun*), *sun* 寸 (1 *sun* = 10 *bu*), *bu* 分 (1 *bu* = 10 *rin*), and *rin* 厘. In this instance, the '0' in '2 *sun* 07' should have *bu* as its units, and '7' as *rin*. However the author Kobayashi Hirokichi instead places *bu* where *rin* should be.

3.3.9.1 Solving with Tenzan Jutsu

Transcription	Translation
今如圖大圓二個中圓四個交觔容小圓四個只云大圓一寸問小圓径幾何	As in the diagram, there are two large circles (大) and four medium circles (中). Inside the space where these circles intersect are four small circles (小). Say the diameter of the large circles is 1 *sun*. Problem - what is the diameter of the small circles?
答日　小圓二分零七毛　有奇	Answer: The diameter of the small circles is 2 *bu* 0 7 *mo* ...
術日　置五分開平方内減分五餘乘大圓径小径得合問	Technique: Put 5 *bu* and take the square root. Inside subtract 5 *bu*. Then multiply by the diameter of the large circles. Obtain the diameter of the small circles as required.

The problem in the *Sanpo Tenzan Shinan* does not have the same issue with labels. In this instance all measurement units are labelled correctly, as we are told the '7' in '2 *bu* 07' is in terms of *mo*. This implies that the '0' is in terms of *rin*, since 1 *bu* = 10 *rin* and 1 *rin* = 10 *mo*.

Given Instructions:

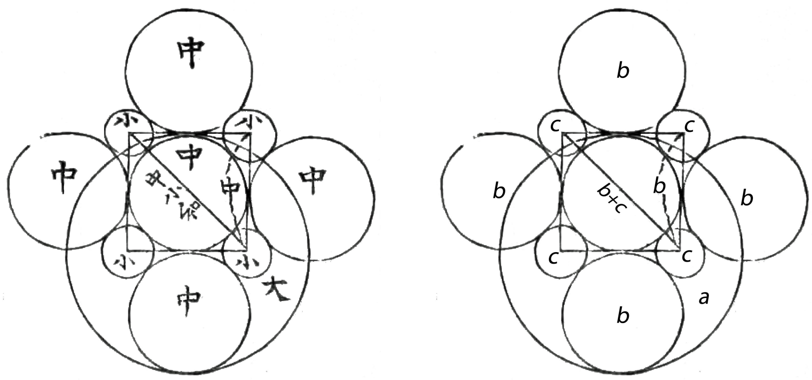


Figure 3.23: Left: Diagram from the *Sanpo Tenzan Shinan*. Right: Transcription.

[Let 大 be represented as a, 中 as b, and 小 as c.]

解日置大半之

Solution: Put 大 [a] and halve it.

二	大	2	a	$\frac{a}{2} = b$
中径		b		

乘方斜率

Multiply by the square diagonal ratio.[38]

二	大二 商	2	a 2 $\surd$	$\frac{a\sqrt{2}}{2} = b + c$
中径和小		$b + c$		

乘除省之

Multiplication and division. Eliminate.[39]

二 商	大	2 $\surd$	a	$\frac{a}{\sqrt{2}} =$ b + c
中径和小		$b + c$		

寄左 ◯ 以中小和相消求

Move left. Combine 中 [b] and 小 [c] and cancel to find.[40]

イ		e		
二 商	大	2 $\surd$	a	$\frac{a}{\sqrt{2}}$

[38]A square is formed Figure 3.23 inscribing the middle circle 中. Each side of the square is the length of the diameter 中. Each corner of the square touches the middle of the four circles 小. In this step, the diagonal - which has a ratio of $\sqrt{2}$ - is multiplied by the diameter of the circle 中, which is the side length of the square. This produces the value of the diagonal.

[39]The denominator of 2 is converted to $\sqrt{2} \times \sqrt{2}$. Then one instance of $\sqrt{2}$ is cancelled out to produce $\frac{a}{\sqrt{2}}$.

[40]The 'cancel' may refer to a cancellation of 小 [c] from the right-hand side.

ロ	ro	
二 | 大	2 | a	- $\frac{a}{2}$
| 小	| c	- $c = 0$
合矩	0	

變換得

Convert to obtain.

イ	e	
| 五大分商	| $5\ a$ bu $\surd$	$a(\sqrt{0.5})$
| 五大分	| $5\ a$ bu	- $a(0.5)$
| 小	| c	- $c = 0$
合矩	0	

放是求得小径式

Request an equation satisfied by 小 [c].

| 五大分商	| $5\ a$ bu $\surd$	$a(\sqrt{0.5})$
| 五大分	| $5\ a$ bu	- $a(0.5) = c$
小	c	

依絶答術則如左

By means of the absolute answer technique [the answer is] shown left.

Remarks

The formula provided on the tablet in modern notation reads as such

$$a(\sqrt{0.5}) - a(0.5) = c \tag{3.12}$$

In the technique section of the *Sanpo Tenzan Shinan* problem, this is expressed by the equivalent $(\sqrt{0.5} - 0.5) \times a$. Using the original terms provided in the text, this reads $(\sqrt{5bu} - 5bu) \times$ 大. However, the tablet problem appears to give the formula $(\sqrt{5} - 5) \times a$, or in the original text $(\sqrt{5ko} - 5ko) \times$ 大, which does not produce the value of 2.07 given on the tablet. It is curious that the author has done this, for when checking his calculations it should have become immediately clear that an error occurs when using *ko* instead of *bu*.

I have discussed this problem with Mitsuo Morimoto, who believes that due to the fact that the formula was supposed to be calculated on the *soroban*, it can be considered vague and ambiguous but not necessarily incorrect. The formula tells us to put down the number 5, which we take the square root of, subtract 5 from, and multiply by 10. On the *soroban* one row represents ones, the next tens, the next hundreds, and so on. This means our 5 has the potential to be expressed 0.05, 0.5, 5, 50, or 500 depending on which column we put it in. On the *soroban*, the user can also choose which column they wish to start from (i.e. use as the ones). The user can also put down a value in any row and treat it as abstract. That is, we can put down 5 and consider it to still represent 5 no matter which column it is placed in. It is possible that due to this vagueness the author placed the value 5 in the ones column but treated it as if it were in the tenths column.

By examining the *Sanpo Tenzan Shinan* example, we can see how a traditional Japanese mathematician could have approached this *sangaku* problem. The solution essentially relies on a recognition of the square and diagonal formed in Figure 3.23. From this it can be determined that $\frac{a}{2} = b$ and $\frac{a}{\sqrt{2}} = b + c$. By substituting $\frac{a}{2}$ for b the value of c can then be obtained.

3.3.10 Nagano Tenman-gū: Second Problem

Translation

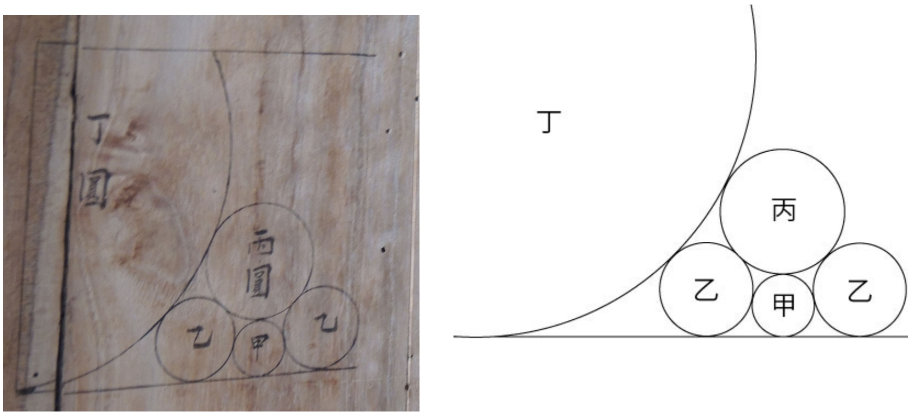


FIGURE 3.24: Left: Second Nagano Tenman-gū Problem. Right: Transcription. (Image by author).

TRANSCRIPTION	TRANSLATION
今如圖直線五圓載有只云甲圓径二寸乙圓径三寸丁圓径幾何問	As in the diagram, there is a line and five circles. Say the diameter of the circle *kō* 甲 is 2 *sun* and the diameter of *otsu* 乙 is 3 *sun*. [Problem -] what is the diameter of the circle *tei* 丁?
答日　丁圓拾八寸	Answer: The diameter of *tei* 丁 is 18 *sun*
術日　以甲圓径二段與乙圓径差乙圓径除自之甲圓径乘問合	Technique: Double *kō* 甲 and subtract *otsu* 乙. Divide into *otsu* 乙 squared, multiplied by *kō* 甲. [Obtain the diameter of *kō* 甲] as required.

3.3.10.1 Solving with Tenzan Jutsu

Once more the same problem appears in the *Sanpo Tenzan Shinan*, with only the labels for two circles being changed.

TRANSCRIPTION	TRANSLATION
今有如圖直線載五圓只云甲圓径二寸乙圓径三寸問丙圓径幾何	As in the diagram, there is a line and five circles. Say the diameter of the circle *kō* 甲 is 2 *sun* and the diameter of *otsu* 乙 is 3 *sun*. Problem - what is the diameter of the circle *hei* 丙?
答日　丙圓径一十八寸	Answer: The diameter of *hei* 丙 is 18 *sun*
術日　以甲径二段與乙径差除乙径自之乘甲径得丙径合問	Technique: Double *kō* 甲 and subtract *otsu* 乙. Divide into *otsu* 乙 squared, multiplied by *kō* 甲. Obtain the diameter of circle *hei* 丙 as required.

Given Instructions:

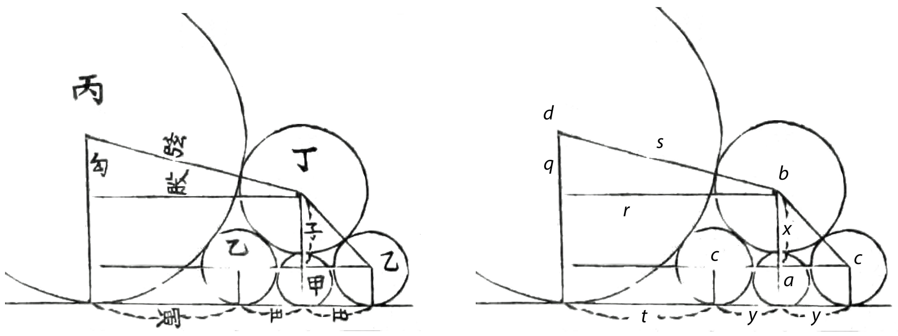


FIGURE 3.25: Left: Diagram from the *Sanpo Tenzan Shinan*. Right: Transcription.

[Let 甲 be represented as a, 丁 as b, 乙 as c, 丙 as d, 子 as x, 刃 as y, 勾 as the short leg q, 股 as the long leg r, 弦 as the hypotenuse s, and 寅 as the line t.]

解日依圖求各

Solution: Using the figure, find each.[41]

二 | 乙	2 | c	$-\frac{c}{2}$
二 | 丁	2 | b	$(+)\ \frac{b}{2}$

[41]From Figure 3.25, the lengths 子 [x] and 刃 [y] are obtained by constructing a right angle triangle abc.

甲	a	(+) a $= x$
子	x	
甲乙 商商	$a\ c$ $\surd\ \surd$	$\sqrt{c} \cdot \sqrt{a} = y$
刃	y	

子冪刃冪相併寄左 ◯ 以乙丁和半冪相消求矩合同加異減得

子 squared [x^2] and 刃 [y^2] combine and put left. Therefore sum of 乙 and 丁 halved, and squared [$\left(\dfrac{c+b}{2}\right)^2$]. Cancel and find equation[42]. Add for the same, subtract for the different to obtain.[43]

甲 巾	a 2	- a^2
甲丁	$a\ b$	- $b \cdot a$
乙丁	$c\ b$	(+) $b \cdot c = 0$
合矩	0	

[42]The Pythagorean theorem is applied to triangle abc to produce $\left(a + \frac{b}{2} - \frac{c}{2}\right)^2 + (\sqrt{a}\sqrt{c})^2 = \left(\frac{c+b}{2}\right)^2$. Then $\left(a + \frac{b}{2} - \frac{c}{2}\right)^2 + (\sqrt{a}\sqrt{c})^2$ is shifted it to the right-hand side. The term 'equation' (合矩) indicates the equation is to be made equal to zero, giving $-\left(a + \frac{b}{2} - \frac{c}{2}\right)^2 - (\sqrt{a}\sqrt{c})^2 + \left(\frac{c+b}{2}\right)^2 = 0$.

[43]As mentioned in section 3.2.2, this rule indicates that when integers have the same sign you add them, and when they do not you find the difference between them. This step is essentially a form of expanding and simplifying, and it produces the form $-a^2 - ab + cb = 0$.

括之

Put together.[44]

甲 巾	a $_2$	- a^2
甲丁 乙 差	c b a —	$(+)\ b(c-a) = 0$
合矩	0	

故求

Therefore find.

甲 乙 差 | 甲 巾	a c — | a $_2$	$\dfrac{a^2}{c-a} = b$
丁	b	

又求各

And find each.

二 | 丙	2 | d	$\dfrac{d}{2}$
二 丁	2 | b	- $\dfrac{b}{2}$

[44]This instructs to simplify further and combine similar elements. Since in $-a^2 - ab + cb = 0$ both $-a$ and c are multiplied by b, we can combine them to form $b(c-a)$.

甲	a	- a = short leg
勾	*short leg*	
丙乙 商商	$d\ \ c$ $\surd\ \surd$	$\sqrt{c}\cdot\sqrt{d}=t$
寅	t	
刃	y	y
寅	t	(+) t = long leg
股	*long leg*	

解刃寅

Splitting 刃 [y] and 寅 [t].[45]

甲乙 商商	$a\ \ c$ $\surd\ \surd$	$\sqrt{c}\cdot\sqrt{a}$
丙乙 商商	$d\ \ c$ $\surd\ \surd$	(+) $\sqrt{c}\cdot\sqrt{d}$ = long leg
股	*long leg*	
二 │ 丁	2 │ b	$\frac{\mathrm{b}}{2}$

[45]Using the splitting rule, substitute the earlier found value of $\sqrt{c}\sqrt{b}$ for 刃 [y] and and $\sqrt{c}\sqrt{d}$ for 寅 [t].

二 | 丙 弦	2 | *d* *hypotenuse*	$(+)\ \frac{\mathrm{d}}{2} = \text{hypotenuse}$

而依弦冪適等求矩合同加異減

And using 弦 squared [*hypotenuse*^{2}] find equal terms in equation. Add for the same, subtract for the different.[46].

イ 丙丁	*e* *d b*	- $b \cdot d$
ロ 甲丙	*ro* *a d*	- $d \cdot a$
イ 甲丁	*e* *a b*	$(+)\ b \cdot a$
ロ 甲 巾	*ro* *a* 2	$(+)\ a^2$
ハ 丙乙	*ha* *d c*	$(+)\ c \cdot d$
ハ 甲丙乙 商商	*ha* *a d c* √ √	$(+)\ 2c(\sqrt{d} \cdot \sqrt{a})$

[46]According to the Pythagorean theorem, what is equal to *hypotenuse*^{2} is *long leg*^{2} + *short leg*^{2}. This is applied to the triangle such that $\left(\frac{d}{2} - \frac{b}{2} - a\right)^2 + (\sqrt{a}\sqrt{c} + \sqrt{d}\sqrt{c})^2 = \left(\frac{b}{2} + \frac{d}{2}\right)^2$. Then expansion and simplification occurs. The use of the *iroha* labels appears here, and they are used to mark terms with common factors to be factorised in the next step.

ハ	*ha*	
甲乙	$a\ c$	$(+)\ c \cdot a = 0$
合矩	0	

而括之

Then put together.[47]

イ	*e*	
甲丁 丙 差	$a\ b$ d $-$	- $b(d-a)$
ロ	*ro*	
甲甲 丙 差	$a\ a$ d $-$	- $a(d-a)$
ハ	*ha*	
甲乙 丙 商 加 巾	$a\ c$ d $\surd$ $+$ 2	$(+)\ c(\sqrt{d}+\sqrt{a})^2 = 0$
合矩	0	

變換之

Convert.[48],

甲甲丁 丙丙 商商 差加	$a\ a\ b$ $d\ d$ $\surd\ \surd$ $-\ +$	$-b(\sqrt{d}+\sqrt{a})(\sqrt{d}-\sqrt{a})$
甲甲甲 丙丙 商商 差加	$a\ a\ a$ $d\ d$ $\surd\ \surd$ $-\ +$	- $a(\sqrt{d}+\sqrt{a})(\sqrt{d}-\sqrt{a})$

[47]Here rather than combining $ab + a^2 - db - ad$ as $a(a-d)$ and $b(a-d)$, the author combines them as $-a(d-a)$ and $-b(d-a)$.

[48]To get rid of the square in $c(\sqrt{d}+\sqrt{a})^2$, the term $d-a$ in $-b(d-a)$ and $-a(d-a)$ is converted to the equivalent $(\sqrt{d}+\sqrt{a})(\sqrt{d}-\sqrt{a})$. Later all terms are divided by $\sqrt{d}+\sqrt{a}$.

甲乙 丙 商 加 巾	$a\ c$ d $\surd$ $+$ 2	$(+)\ c(\sqrt{d}+\sqrt{a})^2 = 0$
合矩	0	

遍省過乗

Using elimination of higher power terms many times.[49]

甲丁 丙 商 差	$a\ b$ d $\surd$ $-$	- $b(\sqrt{d}-\sqrt{a})$
甲甲 丙 商 差	$a\ a$ d $\surd$ $-$	- $a(\sqrt{d}-\sqrt{a})$
甲乙 丙 商 加	$a\ c$ d $\surd$ $+$	$(+)\ c(\sqrt{d}+\sqrt{a}) = 0$
合矩	0	

解丁遍乗除

Splitting 丁 [b] using multiplication and division many times.[50]

甲甲 丙巾 商 差	$a\ a$ $d\ 2$ $\surd$ $-$	- $a^2(\sqrt{d}-\sqrt{a})$

[49]Here the power in $c(\sqrt{d}+\sqrt{a})^2$ is eliminated by dividing all elements by $\sqrt{d}+\sqrt{a}$. This produces $-b(\sqrt{d}-\sqrt{a})-a(\sqrt{d}-\sqrt{a})+c(\sqrt{d}+\sqrt{a})=0$.

[50]Here b is substituted for the value $\frac{a^2}{c-a}$ obtained in the earlier set of calculations to produce $-\left(\frac{a^2}{c-a}\right)(\sqrt{d}-\sqrt{a})-a(\sqrt{d}-\sqrt{a})+c(\sqrt{d}+\sqrt{a})=0$. Then $-\left(\frac{a^2}{c-a}\right)(\sqrt{d}-\sqrt{a})$ is moved to the right-hand side of the equation, and the left-hand side is multiplied by $(c-a)$. $a^2(\sqrt{d}-\sqrt{a})$ is then moved back to the left-hand side.

甲甲甲 丙　乙 商　差 差	$a\ a\ a$ $d\ \ \ c$ $\surd\ \ -$ $-$	- $(c-a)a(\sqrt{d}-\sqrt{a})$
甲乙甲 丙　乙 商　差 加	$a\ c\ a$ $d\ \ \ c$ $\surd\ \ -$ $+$	$(+)\ (c-a)c(\sqrt{d}+\sqrt{a})=0$
合矩	0	

解之同加異減

Splitting, add for the same, subtract for the different.[51]

乙甲 巾商	$c\ a$ $2\ \surd$	$\sqrt{a}\cdot c^2$
甲乙丙 　　商	$a\ c\ d$ $\ \ \ \ \surd$	- $2\sqrt{d}(c\cdot a)$
乙丙 巾商	$c\ d$ $2\ \surd$	$(+)\ \sqrt{d}\cdot c^2=0$
合矩	0	

遍省乙括之

Use elimination of 乙 [c] many times, put together.[52]

乙甲 　商	$c\ a$ $\ \ \surd$	$\sqrt{a}\cdot c$

[51] The previous calculation is expanded and simplified.

[52] This step eliminates c from $-2ac\sqrt{d}$ by converting $c^2\sqrt{d}+c^2\sqrt{a}$ to $c(c\sqrt{d}+c\sqrt{a})$, moving it to the right-hand side, and then dividing the left-hand side by c. Then $c\sqrt{d}$ is moved to the left, and the left is converted to $\sqrt{d}(2a-c)$. All terms are moved back to the left.

甲丙 二商 乙 差	a d 2 $\surd$ c $-$	$-\sqrt{d}(2a-c)=0$
合矩	0	

故求丙商

Therefore find square root 丙 [$\sqrt{d}$].[53]

甲二乙差 | 乙甲 商	a 2 c $-$ | c a $\surd$	$\dfrac{\sqrt{a}\cdot c}{2a-c}=\sqrt{d}$
商丙	$\surd$ d	

自之得

Self multiply to obtain.[54]

甲二乙差巾 | 甲乙 巾	a 2 c $-$ 2 | a c 2	$\dfrac{a\cdot c^2}{(2a-c)^2}=d$
丙	d	

依絶答術則如左

By means of the absolute answer technique [the answer is] shown left.

Remarks

The *Sanpo Tenzan Shinan* produces the following formula in modern notation

$$\frac{a\cdot c^2}{(2a-c)^2}=d \tag{3.13}$$

When rearranging the labels on the tablet and inputting the values in, the answer on the tablet of 18 can be found.

[53] $\sqrt{d}$ is found by shifting $c\sqrt{a}$ to the right-hand side and then dividing the $(2a-c)$ in $\sqrt{d}(2a-c)$ by it.

[54] The square root is elimated by squaring all items.

3.4 Sangaku and Wasan Mathematicians

As seen in the previously examined case studies, *sangaku* can be solved using traditional methods of *wasan*. While this showed how *sangaku* can be solved traditionally, the question remains of how these tablets fit within the broader *wasan* tradition. In this section I argue that *sangaku* constitute traditional problems of *wasan*, and can be seen to connect back to this tradition through their format and language.

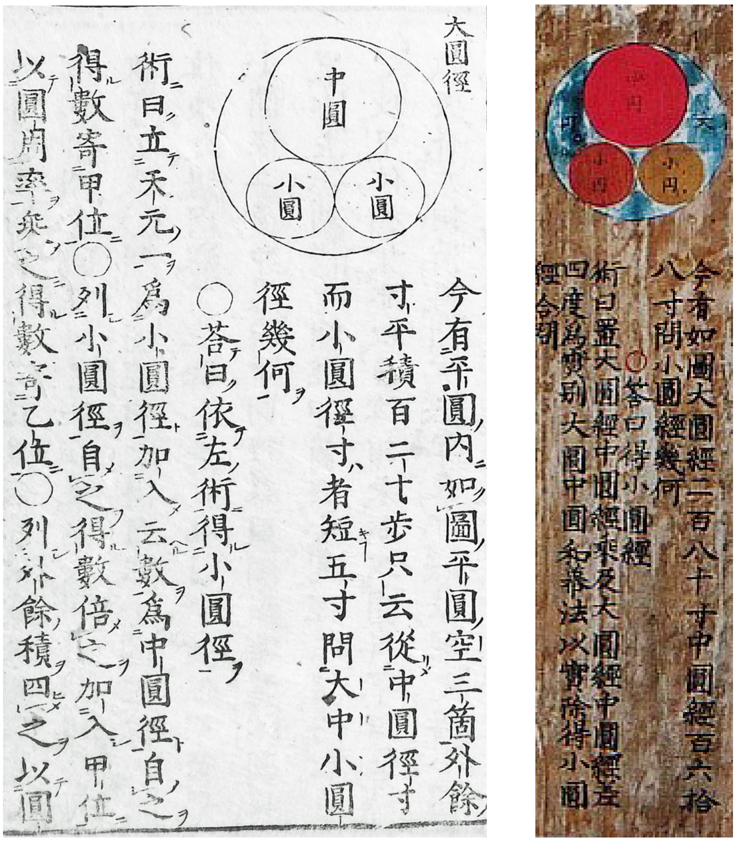


FIGURE 3.26: *Hatsubi Sanpo* of Seki and a *sangaku* problem from Toride Shrine

One way *sangaku* connect with *wasan* is through the use of the same problem, answer, and technique section format as Seki's work. An example of this can be seen in Figure 3.26, which shows a problem found in Seki's *Hatsubi Sanpo* beside a *sangaku* problem from Toride shrine. In this instance, the two problems are identical in that they seek the diameter of the small circles (小) and provide an answer in terms of a formula in the formula section. But not only is the subject matter similar, but the format of the problem into the three problem-answer-technique sections. As mentioned in section 3.2.1, the *Hatsubi Sanpo* often presented problems which did not use numerical answers and instead directed the observer towards the formula section. On the Toride tablet we see an example of this occuring on a *sangaku*, and this style of Seki shows up on many other *sangaku*. For instance, another example is found on the Okiku Inari *sangaku* from section 2.8.10. These examples and the examined case studies show that the format and language style of *sangaku* problems was similar to other problems of the *wasan* tradition.

It is also the case that many prominent mathematicians who came after Seki - both from his own school or rival ones - created and dedicated *sangaku*. An example is the case of the previously discussed Aida Yasuaki. Aida fell out with Fujita Sadasuke 藤田定資 (1734 - 1807), who was the head of the Seki school. Due to this, Aida founded his own school known as the Saijyo. However, the reason for this falling out is believed to be due to a *sangaku*. Kazuo Shimodaira writes

> Fujita did not receive Aida as a pupil, however, perhaps because of a falling-out occasioned by Fujita's pointing out mistakes in the problems inscribed on a tablet donated to a temple by Aida...Aida then devoted his efforts to composing and publishing his *Kaisei sampo* (1781), in which he criticized and revised Fujita's highly regarded *Seiyo sampo* of 1781 [85, p. 83].

The twenty-two tablets created by members of schools linked to the Seki in the Isaniwa shrine of Matsuyama (detailed in 道後八幡 - 伊佐爾波神社の算額) also provides further evidence of highly mathematically educated individuals producing *sangaku* [86]. As well as this, it is known that the *Shinpeki sampo* 神壁算法 (date unknown) text of Fujita was in fact a collection of problems specifically taken from *sangaku*. Fujita had recorded these problems in order to make it easier for mathematicians nationwide to access *sangaku* problems. Horiuchi explains that the first volume of the *Shinpeki Sanpo* exclusively deals with tablets dedicated by members of Fujita's school [31, p. 142].

These examples show how *sangaku* were being created and solved by leading mathematicians with knowledge of *tenzan jutsu*. In fact, *sangaku* were such a part of mathematical practice that the rift between Aida and Fujita - which led to the creation of a whole new school of mathematics - was sparked by a tablet. We also see in this episode that practitoners were aware of the work others had done in the form of *sangaku*, and judged one another's abilities on their tablets. This indicates that the function of these tablets as transmission devices worked, for practitioners were aware of each others mathematical works. Another instance of this is seen in the Isaniwa shrine, where a large number of *sangaku* appear to have been dedicated in response to the placement of other tablets in the shrine.

This shows that prominent mathematicians were creating, reading, and solving *sangaku*. Their content was also similar to work in the wider tradition. Proof of similar content was seen in the previously mentioned case studies, and further evidence can be found by examining some of the books produced by Aida. One example is the *Sanpo Tenshoho Shinan*. This book is described as

> ...a collection of conventional geometry problems which were, however, presented in a new and simplified symbolic notation...Aida compiled the geometry problems presented in Arima's *Shuki sampo* and Fujita's *Seiyo sampo* and *Shinpeki sampo*, These were largely the problems of *yo jutsu*, the inscribing in circles or triangles of other circles, a mainstay of traditional Japanese mathematics. In his book, Aida also showed how to develop formulas for ellipses, spheres, circles, regular polygons, and so on, and explained the use of algebraical expressions and the construction of equations. [85, p. 83-4][55]

The *Sanpo Tenshoho Shinan* - a work on algebraic equations and conventional Japanese geometry - presents many problems similar or identical to those found on *sangaku*. For example, one problem examined as a case study from the Satimiya shrine can be found in this textbook. Also all the case study problems appeared in the *Sanpo Tenzan Shinan*. Therefore the problems of *sangaku* can be placed in the wider tradition as conventional geometry, with *sangaku* having a clear function of "exhibition of scholarship as a supplement to textbooks" [85, p. 83]. In conclusion, many of the problems found on *sangaku* are synonymous with conventional, mainstream problems of the Japanese tradition.

3.5 Concluding Remarks

In this chapter, I have showed some of the ways in which *sangaku* problems can be solved using *wasan* techniques. Some of the methods of *wasan* were discussed, with particular focus given to the technique of symbolic manipulation known to the Japanese as *tenzan jutsu*. I then transcribed and translated solutions to problems similar to *sangaku* which were solved using *tenzan jutsu*. I discussed how these solutions illustrated one way in which *sangaku* problems could be solved using traditional techinques of the Edo period. Through an examination of *tenzan jutsu* calculations given by Ohara Toshiaki in the *Sanpo Tenzan Shinan*, it could be seen how the general method of solving these problems was to employ diagrammatic reasoning, and reason information from the diagram. Usually this involved finding different ways to express one figure or element on the figure, forming an algebraic equation, then solving for the desired term.

This examination also showed the similarity between *sangaku* and general *wasan* problems, to the extent that many *sangaku* problems seem to have been directly lifted out

[55]The word 算法 can be transliterated as either *sanpo* or *sampo*. This is due to the fact that the n sound is sometimes pronounced more like m in Japanese. However because there is offically no character for m in Japanese, I use the translation *sanpo*.

of *wasan* texts. This is perhaps not unexpected considering the evidence given showing *wasan* mathematicians produced and solved *sangaku*. From this I contend that the primary function for *sangaku* was indeed for problems to be presented, noticed, recorded by others, and showed to fellow mathematicians. An additional function however, which I will show in the next chapter, is their public one as *ōema* tablets and public art works.

Chapter 4

Religious and Artistic Functions of Sangaku

4.1 Introduction

Chapter 1 showed how *sangaku* - while appearing to be gifts to the gods visually - were considered communicative and self promotional challenges by authors such as Horiuchi (1998), Suzuki (2001), and Majewskie et al (2010). Others such as Rothman and Fukagawa (2008), Tarnai and Miyazaki (2005), Dyson (2008) and Heeffer (2012) on the other hand believe *sangaku* were works of art in their own right, and combined mathematics, religion, and art.

One reason for considering these tablets as objects of religion and art comes from their origin. As discussed in section 2.3, *sangaku* are believed to have developed out of the *ōema* tradition which emerged during the Muromachi (1333 – 1573 CE) and early Edo periods. They are often described in the literature as "mathematical ema" [36, p. 84], though no detailed investigation has been carried out to examine whether *sangaku* can be considered to possess the same functions as *ōema*. In this chapter, I contend that when examined as objects of material culture *sangaku* can indeed be seen to operate in a similar manner to *ōema* as objects of religious offering and art. However, I stress that this was not their only function, for *sangaku* also clearly possess a mathematical function as exhibited in chapter 3.

I begin by first outlining the material culture methodology I will apply to *sangaku*. I then examine the objecthood, production and consumption of these tablets. From this, I argue that *sangaku* can be considered a sub-family of *ōema* which share the same religious and artistic functions due to their construction with the same materials, location of display in shrines and temples, and their function as offerings.

I contend that while *ōema* combined the sacred and secular realms, *sangaku* combined the sacred, secular, and academic. The mathematical content of these tablets did not hinder their public reception as *ōema*, and the connection *sangaku* had to both these tablets and shrines brought additional religious significance. Therefore *sangaku* can therefore be considered an intersection between the *ōema* and *wasan* traditions.

4.1.1 Material Culture Studies

The study of material culture promotes the investigation of all aspects of historical artefacts. We are asked not only to consider the ideas conveyed or expressed by objects, but to also examine their physical properties. For example, a material culture approach to a Babylonian tablet might ask questions such as what clay and raw materials were used to make the tablet, where the materials were from geographically, who crafted the materials to form the tablet, what processes they used, where the tablet was kept once inscribed with mathematics, who saw it, and why it was created. Such investigations move beyond an examination of the mathematical concepts presented to give a fuller picture of the whole artefact itself as the product of a particular culture and time period.

Viccy Coltman offers the following model for the analysis of artefacts of material culture in *Material Culture and the History of Art(efacts)* [11, p. 44], which I use as a model for my examination of *sangaku*:

Objecthood - What is the object? What does it represent? What is it made of? What size and/or weight is it? What date was it created?

Production - Who made it? Where was it made? What is noteworthy about its construction, design, and technique? Is it the only one?

Consumption - Who owned it? How did they acquire it? If they purchased it, how much did it cost? What was its function? Where was it used and/or displayed?

In the following sections I apply these questions to *sangaku*, and argue that when examined in their entirety as historical artefacts they closely resemble *ōema* and should be considered a sub-family of this Japanese tradition.

4.2 Objecthood

When examining a historical artefact, Coltman asks us to consider what the object is, its physical weight and dimensions, what materials were used to construct it, the date it was created, and what the object was intended to represent.

The first of these appears straightforward. We know from the work of historians such as Fukagawa that *sangaku* were mathematical tablets dedicated to shrines and temples by the Japanese largely during the Edo and Meiji periods. *Sangaku* were also intended to represent mathematical problems which had been solved by the author of the tablet's content. With regard to the other questions posed, a deeper examination is required.

4.2.1 Dates, Dimensions, and Weight

In answer to the question of what date *sangaku* were created, one may recall that in chapter 2 it was discussed that the oldest known surviving *sangaku* dates back to 1683 CE. While there are references to older tablets in the literature, their dates do not fall before the beginning of the Edo period. Further details on dates can be found in section 2.3.

With regard to dimensions, each *sanagku* is different and there is no standard size. In the text 道後八幡 - 伊佐爾波神社の算額 *Dōgō Hachiman-gu - Sangaku of Isaniwa Shrine* [86], a collection of twenty-two *sangaku* from the Isaniwa shrine of Matsuyama are detailed. The dimensions for each tablet are given, and they range from 111.5 cm in height to 126.8 cm in width. This is unfortunately one of the few instances in which the dimensions of *sangaku* are provided in the literature. However some further details can be gathered from Kotera's website `http://www.wasan.jp`. An examination of 250 *sangaku* listed on the website shows the tablets to have an average width of 153 cm and average height of 69.4 cm. The longest recorded height is 178 cm, and the longest width is 486 cm.

4.2.2 Physical Materials

To answer the question of what *sangaku* were made of, the physical materials used for their construction must be analysed. This includes elements such as the wood used for the canvas as well as the paints for their decoration.

4.2.2.1 Paints

To date no investigation of the paints used for *sangaku* have been conducted, and such a detailed investigation is beyond the scope of this thesis. However, in the *Kyōdo Sūgaku Sōsho*, Kōgō Hagino claims that on *sangaku* the "figures of problems and ordinary ema are painted the same way" [27, p. ii]. In fact, due to their similarities, Hagino writes that "in order to get noticed the figures became complicated and eccentric to distinguish them from Ema" [27, p. ii]. This means that *sangaku* and *ōema* were painted the same way, and given this details regarding the paints used for *ōema* can be applied to *sangaku*.

In *Yanagi Muneyoshi no sekai: "Mingei" no Hakken to Sono Shisō*, Shinzō Ogyū writes that the colours on *ōema* included white pigment, yellow earth pigment, red pigment, red earth pigment, and green copper [66, p. 125]. Robertson also notes that the oldest *ema* discovered to date from the Nara Period (710-784) was "made of cypress. . .and featured an ornamented and saddled horse rendered in red and white pigments" [70, p. 70], indicating that pigment paints have long been part of the tradition.

In Japanese art the main colours were red, yellow/brown, blue, green, and white, although purple and orange could be made by mixing other colours [78, p. 27-30]. Colours are grouped under the term *enogu*. There were three kinds determined by their levels of thickness. The first was a thin water colour known as *mizu-enogu*, the second a thicker water colour *dei-enogu*, and lastly a hard colour cake which had to be mixed with water and glue in a similar manner to *sumi* known as *iwa-enogu* [76, p. 22].

These were also popular pigments on *ukiyo-e*, with Fitzhugh's research into Japanese pigments also indicating that shell white, vermilion, red lead, and organic yellow were fairly common [17, p. 37]. Copper green was sometimes used instead of malachite, and red and brown earth pigments also appeared [17, p. 37]. The brown pigments were similar to red iron oxide colors, and Fitzhugh identified in some cases the existence of manganese and hematite particles in these pigments [17, p. 36]. The red earth pigments are also derived from hematite [17, p. 29].

4.2.3 Canvas

While there are few mentions of the materials used for the canvas of *sangaku*, in the *Bōei Nenkan* of the Bōei Nenkan Publication Society, a tablet is referred to as a "hinoki sangaku" [2, p. 486]. This term is also found in the *Kanchobetsu Kampo Shuroku* to describe a *sangaku* [3, p. 161]. Upon questioning what material was used for *sangaku* in Japan, members of the Amori City Office stated that *hinoki* was the prime material for creating *sangaku*, though the more common Japanese cedar known as *sugi* was also sometimes used. *Sugi* was much cheaper than *hinoki*, making it desirable to those who did not have great funds available.

Japanese *hinoki* (*chamaecyparis obtusa*) is a cedar or cypress tree native to the country [100, p. 77]. *Hinoki* is an expensive and highly prized building material, and was used to construct many sacred buildings such as the Grand Ise Shrine dedicated to the most revered of the Japanese gods - the Sun goddess *Amaterasu*. *Hinoki* was an attractive wood because it was a softwood and had a straight grain that made it easy to split [4, p. 50]. There was also "its soft golden sheen, its resistance to rot, and its distinctive resinous smell" [4, p. 50] that made it highly valued in carpentry. They are often grown as *bonsai* - which is an art in which one grows miniature trees - due to their high status in Japanese culture. As noted by Hagino, *sangaku* can also have a "framework of black lacquer coating" [27, p. 68]. Many also contain metal pieces on their frames. An example can be seen in Figure 4.1.

FIGURE 4.1: *Sangaku* at Miwa shrine. (Image by author).

4.3 Production

With regard to production, material culture studies ask us to analyse who made artefacts, where they were made, and what is noteworthy about the production. The mathematical problems on *sangaku* were clearly made by a mathematician, and in a large number of cases their name is included on the tablet. However, there were other parties involved in the creation of these tablets - painters and craftsmen.

4.3.0.1 Painters and Craftsmen

Usually the painter of a tablet does not have their name displayed, meaning there is no way to determine who painted *sangaku*. However in the case of the Yoshifuji Mishima shrine tablet (Figure 4.2), oral tradition in Matsuyama holds that the author of the mathematical problem was also the painter. It is known that this particular mathematician was skilled in many areas, and alongside the *sangaku* hangs a painting of Mount Fuji painted by the same author. Therefore while not explicitly stated, other tablets may have been also painted by the author of the problem, for *samurai* in particular were known for their study of other arts as well as mathematics. Painting and calligraphy were however also arts in and of themselves, with a whole professional class dedicated to them.

While the exact details of who created *sangaku* is unknown, it is known that during the Edo period, there were specific workshops available dedicated to creating *ōema*. As mentioned, according to Hagino *ōema* and *sangaku* were painted in the same way, meaning that the painters and workshops used for *ōema* were likely the same as for *sangaku*. Regarding these painters, Snow writes that in the *Edo kanoko* text, two *ōema* making businesses are listed in the Asakusa district of Tokyo [88, p. 49]. She states that "most makers of large ema. . .are referred to as artists", and four schools in particular - the Tosa, Kanō, Hasegawa, and Kaihō - were the primary schools of *ōema* painting [88, p. 50-1]. To the *ōema* workshop workers and the hired artist, a *sangaku* would likely be approached much like any other tablet destined for a shrine or temple. They would use the same paints and style of painting common at the time; starting first with a skeleton outline in *sumi* ink and then moving to pigment based colours if the commissioner could afford it [97, p. 107]. The same care would be taken with the calligraphy, again done in the *sumi* ink.

FIGURE 4.2: Yoshifugi tablet, Matsuyama, Shikoku Island. (Image by author).

4.4 Consumption

Questions of consumption ask us to look at who owned a material artefact, how they acquired it, how it was displayed, and what its function was. In this section I will approach each of these questions, which are critical to fully understanding the nature of the *sangaku* tradition in its entirety. I first look at ownership and acquisition, before examining the display and use of these tablets.

4.4.1 Ownership and Acquisition

With regard to the ownership of *sangaku*, while these tablets were originally commissioned by a mathematician they were not kept by them. Instead, *sangaku* are found located in Shinto shrines and Buddhist temples throughout Japan. Rothman and Fukagawa state that these tablets were freely given as offerings to these religious locations [29, p. xiv]. For this reason, *sangaku* are often referred to as *votive* mathematical tablets in the literature. For instance Makishita states

> The term Sangaku. . .refers to Ema (votive tablets) on which mathematical problems were written and which were dedicated to shrines and temples [51, p. 598]

However, while *sangaku* were given to shrines and temples, this does not necessarily imply that the tablets were specifically donated as votive objects. But an examination of the text on these tablets reveals that many contain the characters *hōnō* 奉納 (or variations) in large letters above the mathematical content. These words can be translated as “offering” or “worshipfully presented” [14, p. 9], and indicate that the tablets were indeed freely given as offerings to shrines and temples.

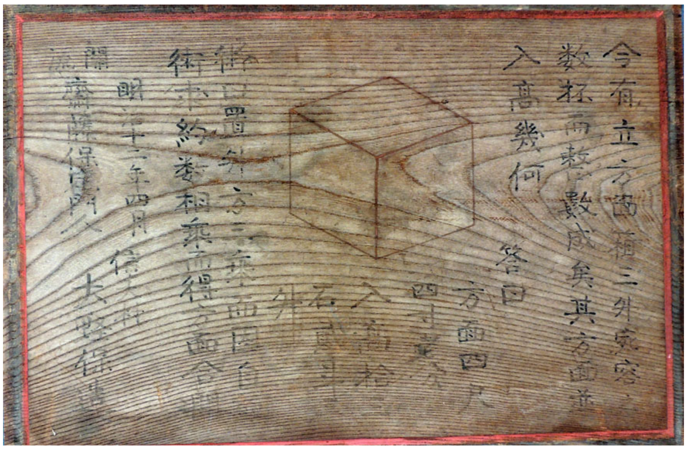


FIGURE 4.3: Kanayama tablet, Iida, Nagano Prefecture. (Image by author).

There are also instances in which the oral tradition of a shrine, temple or area tells of *sangaku* being given to a shrine or temple specifically as an offering. One example of this is the *sangaku* from the Kanayama shrine of Iida in Nagano prefecture (Figure 4.3). Upon visiting the shrine on the 8th of April 2014, I was informed that the oral tradition of the shrine states the author of the tablet - Ono Yasuzō - specifically dedicated it as a form of worship. It is believed to have been dedicated to the specific god of the shrine as thanks for solving the difficult question on the tablet. Ono, it is claimed, considered mathematics to be in the arena of the gods, and to solve a problem was a feat of the gods.

Another example of a tablet created as an offering is the previously examined *sangaku* of the Yoshifuji Mishima shrine (Figure 4.2). Oral tradition holds that the creator Matsuoka was disappointed that his son was not interested in mathematical study. He then created a *sangaku* and dedicated it to the shrine as an offering and plea for help from the gods. A further instance is the Katayamahiko tablet examined in chapter 4. It may be recalled that in the preface for the tablet the author wrote "I dedicate this tablet to the shrine in the hope that my students may get more scholarship in mathematics" [29, p. 145]. Another preface to a *sangaku* by a disciple of Takeda reads

> Mathematics developed from the relations of circles and squares. Mathematics is one of the six educations: manners, music, archery, riding, writing and mathematics...The teacher Takeda has been studying mathematics since he was young. In this shrine, his disciples ask God for progress in their mathematical ability and dedicate a sangaku [29, p. 243].

From these examples, it can be seen that *sangaku* were indeed given as offerings and acts of worship, and ownership of these tablets belonged to the shrine or temple where they were offered.

4.4.2 Display

As mentioned, *sangaku* are traditionally found within Shinto shrines and Buddhist temples throughout Japan. They are generally found either on the eaves of the main shrine or temple hall, or located within the *emaden* or *emadō* 絵馬堂, which was a specific outdoor area designed to store and display *ōema* [69, p. 30]. Further details on the location of these tablets can be found in section 2.4.

4.4.2.1 Textual and Graphical Display

As well as investigating where *sangaku* are physically displayed in shrines and temples, the display of their content is also important to note. In chapter 2 the typical layout of a *sangaku* was described. Usually each problem is broken down into six main sections - diagram, problem, answer, technique, year, and author and school.

Sangaku, as discussed in section 2.7, are usually written in the academic *kanbun* language. However, similar to how English can be handwritten either in a clear printed form or a cursive manner, *kanbun* characters can be displayed differently through the use of different calligraphy styles. The different forms of calligraphy used in Japan were the seal script *tensho*, scribe's script *reisho*, block script *kaisho*, semi-cursive script *gyōsho*, and cursive script *sōsho* [80, pp. 13-14]. Examples of the block, cursive, and seal scripts can be seen in Figure 4.4.

Figure 4.4: Different Japanese Calligraphy scripts. (Image by author).

The oldest of these scripts is the seal script. Markus Sesko explains that the "Seal script is an ancient style of Chinese calligraphy. . .Since *tensho* is an unusual, archaic script with richly graphic features, it has been popular with calligraphers" [83, p. 7]. Over time, other forms of Japanese calligraphy have developed from this script.

Figure 4.5: *Hōnō* 奉納 in seal script on the Kubodera tablet. (Image by author).

With regard to *sangaku*, tablets can be found which use the block style, semi-cursive style, and cursive script. This indicates that there was no set style used, but it is the case that the most common forms appear to be the block and semi-cursive scripts. There are also instances in which the previously mentioned characters *hōnō* 奉納 are written using the Japanese seal script and cursive script. An example of seal script used on a *sangaku* can be seen in Figure 4.5 from Kubodera in Nagano prefecture. Robert W. Gunn writes that "Japanese calligraphy is much more than writing; it must be approached as one would any piece of art" [25, p. 129], indicating that the use of the cursive and seal scripts for *hōnō* 奉納 may have had a specifically aesthetic function.

With regard to graphical display, the diagrams which accompany the mathematical text are key elements. A full discussion of their use on *sangaku* is discussed in chapter 5. Additional to these diagrams, there are instances in which *sangaku* also have graphical scenes. For example, Figure 4.6 shows a scene from a *sangaku* in which mathematicians appear to be working on problems using the *sangi* and *soroban* abacus. One might be mistaken in considering this a standard *ōema* given the heavy focus on the painted scene involving people and everyday life, as it is only the problems painted at the very top which give it away as a *sangaku*. There are further examples such as this, though the majority of *sangaku* do not have such elaborate scenes.

FIGURE 4.6: A *sangaku* in Sozume shrine with a painted scene showing mathematicians along with geometrical diagrams (1861). (H. Kotera[1]).

4.4.3 Audience

In chapter 3 it was seen how *sangaku* were used to transmit mathematical problems amongst mathematicians. This method was successful, for there are cases in which mathematicians did respond to the dedication of a *sangaku* at a shrine or temple by attempting its problem(s). In the case of Aida Yasuaki and Fujita Sadasuke, a great feud was even created due to Aida finding an error in the tablet dedicated by Fujita.

While mathematicians were a key audience, they do not appear to have been the sole one, for there are recorded instances where mathematicians created *sangaku* with the intention to get local people in the community talking and asking questions about their work. Two examples of this are noted in the travel diary of the mathematician Yamaguchi Kanzan (1781 - 1850), which tells of two mathematicians who write "we

[1] See http://www.wasan.earth.linkclub.com/okayama/jyuku.html

decided to hang a *sangaku* in this shrine. We hope that the visitors will look at this tablet and ask for any opinions about the problem" [29, p. 247], and "I want to hang a *sangaku* on which new problems and their solutions are written. If visitors would look at my *sangaku*, then I would be very happy" [29, p. 264]. In these two instances, it appears that the intention of the author was to get the general public to see and engage with their work, meaning that *sangaku* were intended for viewing by a wider audience than just mathematicians. Although non-mathematicians might not be able to read, understand, or transmit the work, they were included in the category of visitors desired to see it. Their location in shrines and temples is also very telling of the audience. Snow writes

> During the Edo period, religious sites became more obviously sites of cultural, not just religious, activity. Pilgrimage to temples and shrines was the first permitted form of widespread tourism in Japan...Ema halls were among the sites local and regional visitors could see while stopping at a temple or shine...Ema halls were unique within the display culture of the Edo period because they allowed anyone to view original paintings and anytime [88, p. 64].

Regardless of social standing, gender, and age anyone in Japanese society could view the paintings dedicated to shrines and temples, as they were public works of the community. As Fisher writes, by being public works they took on additional features and functions, for

> [T]he act of placing or performing a work of creative expression in public space alters how that space is seen, and how audiences see the work; if sufficiently noticed and engaging, it may also alter the ways in which both artist and audience see themselves and their worlds [16, p. 44].

The very placing of a *sangaku* in a public place such as a shrine or temple meant its audience included everyone in the community. Local or visiting non-mathematicians may have seen it merely as an *ōema*, and enjoyed the bright colours and complex diagrams, while mathematicians could enjoy the problems. Snow writes that "The art of donating an ema may still have had a primarily religious connotation for some donors, but viewing them on display in ema halls was a secular and social activity for the general public" [88, p. 151]. Therefore regardless of the original purpose of *sangaku* as a method of transmitting or promoting mathematical work, their audience was varied and included all members of Japanese society.

4.4.4 Use

With regard to the use of *sangaku*, there does not appear to be a clear singular use of these objects. Their location within shrines and temples seems to indicate that they were used as *ōema* by these places, and language such as *hōnō* 奉納 also indicates a use as religious offerings. However, certain individuals also used these tablets to promote and share mathematical problems as discussed in chapter 3. In this section, I examine these factors and the connection between *sangaku* and *ōema*, and argue that *sangaku* form a sub-family of *ōema*.

4.4.4.1 Sangaku as Ōema

Ōema, as will be recalled from section 2.3, are large *ema* tablets dedicated to shrines and temples. They contained paintings of historical scenes or important people, and were considered communal art by the local community. Snow writes that

> [E]ma are painted directly onto board. The boards are then surrounded with heavy lacquer frames that are usually black. The frames typically have metal work fitted at each corner and at the midpoint of the frames. . .they are religious objects as well as art objects and exemplify the mingling of the sacred and secular realms in Early Modern Japan [88, p. 2-3].

These tablets were donated by members of the community as religious offerings. The language of *ōema* was the academic *kanbun* script, and many also began with the characters *hōnō* 奉納. They often functioned as a means to show gratitude, or to pray for a wish to be fulfilled. For instance, there are examples of tablets where bowls of sake are displayed - a potential wish from the donor that they may be able to abstain [14, p. 6]. This practice of dedicating tablets for wish fulfilment and gratitude continues in present day Japan, with smaller *ema* tablets for wishes and thanks a commonplace sight in shrines and temples. Students regularly present a small *ema* to pray for good exam results, and those who are ill pray for their health to improve. The practice thus remains engrained in Japanese culture.

Ōema are often characterised by their vibrantly coloured displays of scenes and people. The tablet in Figure 4.7 from Kitano Tenman-gū shrine, for example, shows what appears to be a battle between *samurai* on horseback and in boats. This style is typical of the *ōema* tablet tradition. Kageo Muraoka and Kichiemon Okamura believe that *oema* fall under the category of *minga* or folk art and should be considered examples of applied art [57, p. 93]. This is also claimed by Amaury Saint-Gilles who writes "Ema are literally an art of the common folk" though there were some "created by artistic masters of their times" [73, p. 157]. It is the believed by Foxwell that some of these masters even used this medium to improve "their reputations as painters" [19, p. 230] by producing particularly compelling *ōema*. This indicates that *ōema* functioned both as objects of religious offering and folk art.

Sangaku and *ōema* share many undeniable similarities, one of which is size. Robertson notes that "Large ema measure up to and over one yard in length and width" [70, p. 48] which is around 90 cm. Snow also discusses an *ōema* which is 118 by 168 cm in her investigation of these tablets [88, p. 11]. This is comparable in size to *sangaku*, which as noted in section 4.2.1 have average dimensions of 69.4 by 153 cm. The material used for the canvas was also the same. As previously noted by Robertson, the oldest *ema* discovered was made of cypress, and Winter writes that "ema were often on cypress (hinoki, Chamaecyparis obtusa)" [100, p. 77]. Also, as mentioned by Snow, *ōema* had "heavy lacquer frames" with "metal work fitted at each corner and at the midpoint of the frames" [88, p. 2-3] similar to what is observed on some *sangaku*.

FIGURE 4.7: An *ōema* in the *ema* hall of Kitano Tenman-gū shrine. (Image by author).

With regard to production, Hagino claims *sangaku* and *ōema* were painted in the same way, and the similarity in style between them is believed to have been one motivation for the development of ever more increasingly complicated diagrams on *sangaku*. As well as this *ōema* use the same forms of calligraphy and display the words *hōnō* 奉納 in a similar manner to *sangaku* using highly artistic forms of calligraphy such as the seal script.

When examining both *ōema* and *sangaku* from a material culture point of view independent from their textual content, they appear to be the same physical objects, as they share the same production methods, canvas, language and function of offering, and display location. It is thus only the mathematical content of *sangaku* which causes them to differ. As *ōema* pre-date the *sangaku* tradition, it is *sangaku* which developed out of the *ōema* tradition rather than *ōema* out of the *sangaku*. For these reasons, I believe it is valid to call *sangaku* votive mathematical tablets or mathematical ema, as they form a sub-family of the *ōema* tradition. I argue that they therefore did have religious and artistic functions, for they were part of a shrine tablet tradition steeped with history and religious significance deeply ingrained in the Japanese culture. For the use of the same language ("offering"), paints, painters, wood, and craftsmen as *ōema* indicates that to the non-mathematician observer *sangaku* would have been seen as almost indistinguishable from *ōema*. To many - such the artists and craftsmen involved and the shrine or temple where it was placed - they were also specifically treated as *ōema*. Therefore the claims by Rothman that *sangaku* were "acts of worship, thanks to the gods for being able to solve a difficult problem" [29, p. xvi-xvii] and Dyson that these tablets are "a work of art as well as a mathematical statement" [29, p. x] are valid.

4.5 Additional Religious Elements

As well as being a sub-family of the *ōema* tablet tradition, *sangaku* are also connected to religion and culture through their subject matter and the type of diagrams they often used.

4.5.1 Landscape Problems

Some examples of *sangaku* with clear religious aspects and subject matter are those which take the form of landscapes and seek to show the height of local mountains. An example can be seen in Figure 4.8, where a *sangaku* from Mizuho shrine in Kijimadaira is displayed. This tablet - created in 1829 - looks west towards Nagano city over the plain where the town of Kijimaidara is located. This particular *sangaku* was created by a mathematician who had been calculating the distance of the shrine from the top of *Kousha* mountain 高社山, Madarao mountain 斑尾山, and to Iiyama city 飯山 - the town across the river from Kijimadaira. According to the oral tradition of the shrine and the town (as told to me by the shrine priest and members of the Kijimadaira Village Office), the local *Kousha* mountain was considered highly sacred by the townspeople, and for this reason it was an object of study of mathematicians in the area. It is also the case that another *sangaku* is found nearby Nagano Tenman-gū which also calculates the distance of this mountain from that particular shrine. These are just two of many *sangaku* which calculate distances to or heights of sacred mountains.

FIGURE 4.8: *Sangaku* at Mizuho Shrine. (Image by author).

To understand this practice, one must first look at the Japanese Shinto religion. Shinto 神道 - literally ‘the way of the gods’ - is the native belief system of the Japanese. It is an animistic religion with a multitude of different gods known as *kami.* Many of these gods are enshrined or worshipped in religious sites known as Shinto shrines. These gods included physical objects such as rocks, trees, waterfalls, and mountains. Andrews explains that

> Because everything, all of nature, contained spirit, everything had power and vitality. The Japanese had a close relationship with their nature gods and believed them to be ancient ancestors. The Japanese built shrines to their ancestor gods throughout the land – to the spirits of waterfalls and fountains, to the spirits of trees and mountains [5, p. 107].

In particular, as Andrews notes, the Japanese “recognized numerous mountain deities and revered many mountains themselves as sacred” [5, p. 107]. These deities had their own category in the religion - *yama no kami* (mountain gods). The religious association with mountains such as *Kousha* caused them to become important features in the local landscape. This prominence also made them objects of study for mathematicians, for they were well known to themselves and the community. While these surveying *sangaku* problems evidence the artistic form that tablets can take - with a mountainous scene laid out before the observer - they also have as their subject matter objects of religious contemplation. This is observed not only in the calculating of the distance of objects considered to be sacred gods, but also in the place they calculate the distance from, which is always a shrine or temple where gods are enshrined or worshiped.

4.5.2 Circular Problems

While *sangaku* which survey the landscape and find the heights of local gods show subject matter directly connected to religion, it can be argued that other problems are indirectly connected to religion through the type of geometry they deal with. In particular, problems with a heavy focus on circles may have such a function.

In Rothman and Fukagawa's *Sacred Mathematics: Japanese Temple Geometry*, 70 problems from their collection of 90 (taken from various *sangaku*) contain circles. Of these, 67 are used in the problem. An examination of 27 individual tablets dating between 1701 and 1914 also reveals 20 containing depictions of circles.

In the Japanese tradition circles have at times historically had religious and cultural significance. For instance, a circle plays a key role in the creation myth of the gods and the Japanese islands, and a circular object is part of the central myth of the sun goddess *Amaterasu* 天照 who is often associated with a circle and claimed to be the ancestor of the Japanese imperial family. Many ritualistic stone patterns in the form of circles (believed to have been for community ritual and burial) have also been unearthed by archeologists which date back to the Middle and Final Jōmon periods (3000 – 300 BCE) [42, p. 235].

Circles were connected to myth and religion in Japan, and were referenced with religious connotations in the creation myth of the *Nihongi* or *Nihon Shoki* 日本書紀 (the second oldest chronicle of Japanese history dating back to 720). In this myth, the two gods who embody the female and male (or yin and yang) in the Japanese tradition (*Izanami* and *Izanagi*) meet by walking in a circle [46, p. 156]. They did so as an act of courtship, and through walking in this circular manner produced offspring who became the mythological gods and Japanese islands. The circle enters into mythology again in the myth of the Sun Goddess *Amaterasu* who was the daughter of *Izanami* and *Izanagi*. *Amaterasu* has traditionally been a particularly important deity for the Japanese people, and is considered "the chief divinity of Shinto" [56, p. 71]. It is believed the mythical first emperor *Jimmu Tenno* 神武天皇 (whose supposed rule was from 711 – 585 BCE) was the great-great-grandson of *Ninigi* 瓊瓊杵, who was the grandson of *Amaterasu* [34, p. 32]. Due to this supposed direct ancestry with *Amaterasu* the Japanese imperial family were considered divine up until after World War II.

The circle is very much associated with *Amaterasu* in the Japanese tradition. For example, she is referenced and embodied "in the simple circle on the Japanese flag, which represents the mirror that is central to her myth" [56, p. 71]. In this myth, the mirror known as *Yata no Kagami* 八咫鏡 was believed to be "the device by which Amaterasu-o-mikami was lured from her cave" after she had hidden herself away and caused darkness in the world [6, p. 216]. The *Yata no Kagumi* mirror is thought to house the very spirit of *Amaterasu*. While being sacred due to this association, it also gained much attention and admiration as a magical object after it was found "miraculously unscathed" after a fire in 960 and survived two more in 1005 and 1040 [34, p. 81-2]. The *Yata no Kagumi*, due to these myths and its connection to *Amaterasu*, "forms part of the Japanese imperial regalia" along with a sword and jewel [6, p. 216]. It is the case that "Of these, the mirror is considered very sacred" to the imperial family [35, p. 10]. Mirrors were also venerable more generally in Japanese society and considered "a mystic symbol of purity" [35, p. 10]. They were thought, for example, to be capable of warding off evil spirits and illness because of a "belief that evil destroys itself on recognising itself" [35, p. 10]. From this, it can be seen that circles were intimately connected to mythology, religion, and the imperial line in Japan. The use of a circle to reference *Amaterasu* on the Japanese flag for example indicates the importance of the circle as a religious symbol. It also shows how circles were used to visually reference aspects pertaining to religion and cultural mythology in the country.

FIGURE 4.9: *Sangaku* presenting circle packing.

Problems which feature circle packings in particular may connect to Shinto and Buddhism. In *Circle Packings and the Sacred Lotus*, Tibor Tarnai and Koji Miyazaki argue for a connection between Buddhist representations of the sacred lotus and the circle packing problems with appear in the *sangaku* tradition. They give examples from the Japanese tradition of circle packing problems that appear for 5, 7, 9, 10, 12, and 18 circles. Figure 4.9 also displays some examples from *sangaku*. In Buddhism, the lotus flower represents purity, growth, enlightenment, and transformation, and it has often been represented in Buddhist art in the form of circle packings due to the fruits of the the lotus forming patterns of circles. As well as appearing in Buddhist art, Tarnai and Miyazaki also note the prevalence of circle packings in Japanese family crests [94, p. 146]. They contend that "the circular pattern of lotus receptacles should be considered the origin of the most significant cultural shapes in Japan" [94, p. 149]. Because *sangaku* appear in Buddhist temples as well as Shinto shrines, the circular imagery found on them - in particular the use of circle packings - can be connected back to religious and cultural aspects of these locations.

Thus the overwhelming focus on circular problems in the *sangaku* tradition may possibly be due to the cultural and religious significance of circles in Japanese society. For as well as being inspired by *ōema* tablets, the creators of *sangaku* may have further attempted to connect their work to religion and mysticism - and thus make them appropriate for placement in shrines and temples - by using circular problems. It may have also been the case that circles were used simply due to their mystical, sacred, pure, and magical connotations. However, it is the case that some *sangaku* did not contain circles. In these instances the *sangaku* usually contained colourful painted scenes of religious or cultural significance that gave them an even more similar appearance to *ema* tablets. Nonetheless, the majority of *sangaku* contained circular imagery, meaning that it seems for some practitioners one of the ways in which their work was responsive to context may have been through the use of religious and culturally significant circular symbolism.

4.6 Concluding Remarks

Through examining how *sangaku* are a form of *ōema*, it has been shown how they encompass the same functions as religious and folk art works of the time. They were collaborative works, and to each person involved in their creation and dedication they could be seen differently. To the carpenter and painter they were another *ōema* on which their work was to be exhibited. To the shrine or temple where they were displayed, they were a religious offering to the gods they worshiped and enshrined. For the public, they were art works found in the gallery style *ema* halls to be viewed as tourist spots and admired and discussed. Even for the mathematician they were more than just a way to promote themselves and show off their work, for they had their problems specifically constructed as *ōema* rather than just having it put into book or manuscript form and circulated. They also included non-mathematicians in their audience who could not always read and solve the mathematical work, and associated shrine tablets with art and religion.

For these reasons, *sangaku* should be considered to function more than just as a promotion and transmission device, for when considering those involved in their construction, display, and reception, it can be seen that they also had in fact religious and artistic significance, which was influenced by the ingrained artistic and religious expressions associated with their construction and display.

Chapter 5

Role of Diagrams

5.1 Introduction

Throughout history, many cultures have employed diagrams as pedagogic and heuristic tools. They accompany a diverse range of mathematical problems, discoveries, and proofs, providing additional - and sometimes necessary - visual explanation and reasoning. Their specific role in historical expressions of mathematics has been explored much in the last decades by authors such as Manders (1995), Netz (1998, 2005), and Saito and Sidoli (2012). In the *sangaku* examined in chapters 2 and 3, diagrams played an important role, with their geometrical properties used to find solutions. While diagrams are a key element of *sangaku*, to date their features and significance are yet to be investigated by historians.

In this chapter, I give a brief examination of the role of diagrams in the *sangaku* tradition. I follow the example of historians of ancient Greek mathematics, and apply their themes of discussion to *sangaku*. I begin by first tackling the question of whether *sangaku* diagrams are drawn such that they satisfy the geometrical relations specified in the problem text. I argue that upon inspection, *sangaku* diagrams are not always metrically precise and are very rarely drawn to scale. The written text of *sangaku* too is often under-determined and requires the diagram to make sense. Additionally, I show their diagrams are also dependent on the text. Following this I discuss the question of whether diagrams had an ornamental or aesthetic function, and show some examples of such a function. I also examine the ways in which diagrams can be considered pedagogical or mnemonic devices. While focus is placed on *sangaku* diagrams, I briefly mention diagrams found alongside explanations and calculations of mathematical problems in published books such as the *Sanpo Tenzan Shinan* with regard to pedagogy. I then question whether there is any evidence that diagrams were the main focus of the author, and created before the text.

5.2 Approaching Sangaku Diagrams

In the history of mathematics, most investigations of historical diagrams have been produced by authors focusing on Greek works. In particular, significant research has been conducted on the relationship between diagrams and proofs. A key author in this area is Reviel Netz, whose 1999 work *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History* exposed the interplay between text and diagram in Greek proofs, creating "a renewed interest in the relationship between the argument in the text and the figure that accompanies it" [75, p. 135] amongst historians. Netz claims that "the diagram is a necessary element in the reading of the text" [59, p. 32], but states that while there are "several ways in which diagram and text are interdependent" [59, p. 39], there is a mutual dependence between them that is necessary for the transmission of mathematical concepts.

Other historians - such as Saito - have also identified other areas of interest with regard to the examination of historical diagrams. For instance, Saito cites some of the key characteristics of Greek manuscript diagrams as being overspecification and indifference to metrical accuracy [74, p. 8]. Netz also discusses metrical accuracy and proportion in diagrams, stating that "The most significant question from a mathematical point of view is whether the diagram was meant to be *metrical*: whether quantitative relations inside the diagram were meant to correspond to such relations between the objects depicted" [59, p. 18].

From the research of Netz and Saito, the following list of themes with regard to historical diagrams can be constructed:

1. Overspecification
2. Indifference to metrical accuracy
3. Determination of objects through the diagram
4. Interdependancy of text and diagram

This list of themes, though used in investigations of Greek diagrams, can also be used as a guide for examining *sangaku*. In addition to these themes, there is another which I believe is relevant to *sangaku* - aesthetics and ornamentation. Unlike Greek diagrams, *sangaku* were beautifully coloured at great effort and cost. As colour and art play an important part in this tradition, I will briefly examine these features also.

5.3 Overspecification

With regard to Greek diagrams, Saito writes that "there exists a strong tendency to draw rectangular and symmetric diagrams even if this is not obligatory" [74, p. 8]. This is what is known as overspecification. This issue is largely found in diagrams that have been reconstructed and transmitted through the manuscripts of later copyists. For instance, one example discussed by Saito and Sidoli (2012) is found in the historian Heiberg's *Bodleian 301* and *Vienna 31* manuscripts. In his recreation of Greek diagarms, Heiberg uses overspecification, and draws certain lines in diagrams as parallel when they were not originally so in the Greek originals.

As the diagrams on *sangaku* are originals, this tradition does not suffer from the same transmission issues. However, there is a tendency for *sangaku* diagrams to be symmetrical even though it is not required by the text. But it is usually only through symmetry of the diagram itself that this is found, with the representation of geometrical figures themselves being true (that is, rectangles represented by rectangles and not squares, and so on). For example, in the second problem on the Suwa tablet (see section 2.8.13.4) shown in Figure 5.1, the problem only requires one smaller circle to be present in the diagram to find the solution. However, an additional five circles are present to give symmetry. Further examples were figures are included to create symmetry in this thesis include the second Enman-ji problem (section 2.8.6.4), the Atago problem (section 2.8.7), the Isaniwa problem (section 2.8.9) as well as other problems on the Suwa (section 2.8.13) and Okiku Inari (section 2.8.10) tablets. Due to this, the problem of overspecification is not a large issue for the *sangaku* tradition.

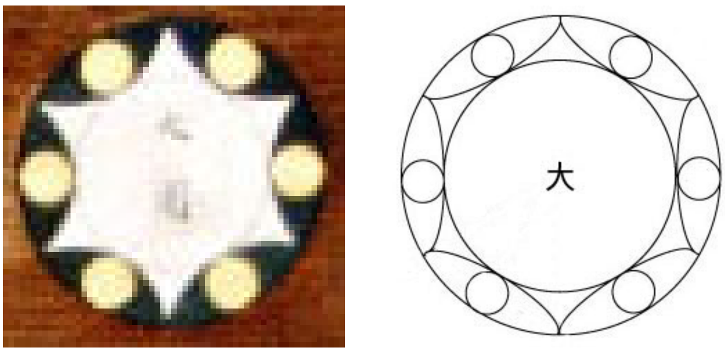


FIGURE 5.1: Left: Second Suwa problem. Right: Transcription. (H. Kotera[1]).

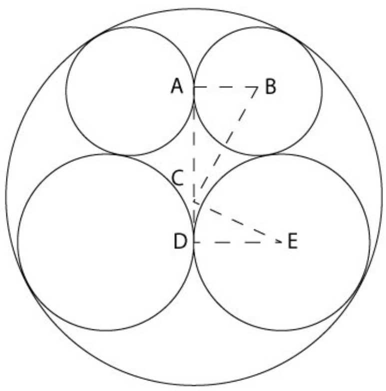
[1]See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

5.4 Metrical Accuracy and Proportion

As mentioned, an issue raised by Saito is that of metric accuracy with regard to diagrams. That is, do diagrams accurately present the mathematical relations occurring in the text? This is noted by Poincaré in his *Analysis Situs*, where he states

> It is worth repeating that geometry is the art of reasoning from badly drawn figures; however, these figures, if they are not to deceive us, must satisfy certain conditions; the proportions may be grossly altered, but the relative positions of the different parts must not be upset. The use of the figures is, above all, then, for the purpose of making known certain relations between the objects that we study [68, p. 6].

For Poincaré, it is not so important for figures to be in the right proportion, but what is vital is that their relations and positions correlate to those intended to be expressed. For example, if a problem relates to two touching circles which lay on a straight line, the diagram should not be drawn such that the circles do not touch one another or the line. It does not matter what size the circles are, and even if they are the size described in the text, as long as they touch one another and the line.



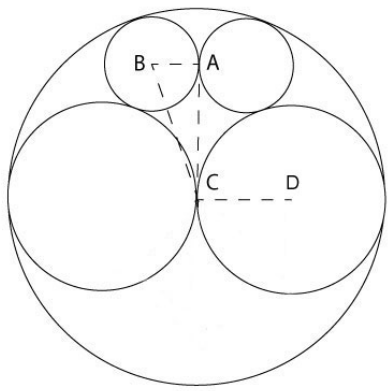


FIGURE 5.2: Left: Diagram showing the original proportions in the diagram for the fifth Suwa problem. Right: Recreated diagram reflecting the proportions given the numerical answer.

In the case of *sangaku*, often the diagrams will grossly misrepresent the proportions of the figures given in the text. Some illustrations of this can be seen on the Suwa tablet (section 2.8.13). But while some problems have measured lengths larger or smaller than what is specified in the text they generally maintain the proportions necessary to illustate the relations for solving the problem. However, there are some instances where the relative proportions are different. For example on the Suwa tablet, some figures are not presented with the correct ratios that should result given the quantitative values of the text. In these instances, the alteration of the proportions causes the relations between the figures - and thus the reasoning required to find the solution - to differ.

For example, in the case of the fifth Suwa problem, in the diagram the author provides - shown left in Figure 5.2 - the problem can be solved by creating two right angle triangles $\triangle ABC$ and $\triangle CDE$ connecting the centre of the outer circle with the centres of the small and medium sized circles. The base of these triangles forms a line AD that constitutes a section of the outer circle diameter. However in the diagram produced using the values from the text - shown right in Figure 5.2 - the medium circles have a diameter equal to the radius of the outer circle (not so in the diagram provided by the author), meaning the right angle triangle $\triangle CDE$ connecting the centre of one medium circle with the centre of the outer circle cannot be formed. The base of the triangle is just a point at the centre of the outer circle, and does not form the line CD. Thus the diagram created from the values in the text requires different reasoning based on the fact that the relationships between the circles in the diagram are different. For in the rightmost diagram, the relationship between the medium circles and the outer is altered, as the medium circles are equal to the radius of the outer. The diagram on the tablet does still enable the reader to see the way to find the solution however.

This evidences that a lack of metric precision did occur in the Japanese tradition. The Suwa tablet in particular also has many examples of figures being represented bigger or smaller than they should be in the text, though the relations are intact. One reason for this may be that while the author constructed the problems for the tablet, the *sangaku* was physically constructed by an *ōema* artist. Because of this, it may be that the artist painting the diagrams on the tablet did not paint them with accurate proportions. Another possibility is that the authors wished to point to a recognition of the general methods which underlied the calculations. That is, when a general method is used it should not matter if the figures were accurate in proportion on the diagram if their configuration and relation to one another was set. Thus by giving circles with different proportions the author may hint that a general method is being used. While for the fifth problem of the Suwa tablet this led to a slightly different approach being used, for other problems the same method can be applied regardless of the drawn size of the figures.

With regard to scale, when examining the Nagano Tenman-gū tablet (section 3.3.8) the diameter of the largest circles were 1 *sun* (roughly 3.03 cm). This is a value which could potentially be exact - that is, the circle drawn on the wood may be exactly 3.03 cm in length - as the tablet is 45 x 45 cm. However, on the second diagram, which is drawn only slightly larger than the first, the largest circle has a value of 18 *sun* or 54.54 cm, which is larger than the entire tablet itself. Another occurrence is seen on the Satimiya tablet (section 3.3.1), where one circle is 36 *sun*, equivalent to 1 m 08 cm. The tablet is only 45 cm in height, meaning the size of the circle should be metrically over twice the height of the tablet, but is visually drawn significantly smaller. Also examining the Isaniwa tablet (section 2.8.9) shows the pictured diagram does not match the values given in the text. The height of the tablet is 56 cm, and the large circle on the diagram converts to 51.813 cm. Again if the measurements were exact, the circle would take up nearly the entire tablet, while at present it takes up less than half this. This is also evident with the Suwa tablet, where each outer circle painted on the tablet has roughly the same size but in the text their sizes vary. This indicates that often diagrams were not drawn to a 1:1 scale on their wooden medium.

While diagrams in the *sangaku* tradition are sometimes out of proportion, they largely fulfil their function of presenting general impressions of relationships between figures which can then be reasoned from to solve the problem presented. This shows that *sangaku* geometry is in a very broad sense an art of reasoning and problem solving from imprecise figures. But though not well presented and precise, in general their function is fulfilled. However, as evidenced there are some cases contrary to this.

5.5 Diagram and Text: Determination and Interdependency

The question of whether objects mentioned in historical texts are adequately determined by their accompanying diagrams, as mentioned, has been discussed much by Netz (1999). That is, do accompanying diagrams provide information necessary to understanding the text, such that if they were not included the content of the text could not be properly established? For example, Netz writes in relation to diagrams that if there is a circle which contains a centre point labelled a, and the text accompanying it states “There is a circle with centre a”, then the point a can be considered fully determined by the text [58, p. 33]. But, if the text reads “There is a circle with radius bc” then it can be considered under-determined, for which of the labels bc mark the points in the centre of the circle and on the circumference is not adequately established [58, p. 33]. In this case the text requires a diagram with a circle of the radius described, with the two points labelled by bc, in order to be fully determined. This issue of determination has been an area of interest for historians of ancient Greek mathematics, and in this section I aim to illuminate whether *sangaku* find determination of their objects through diagrams, and if there is a similar interdependency between the diagram and text as found in the Greek tradition.

5.5.1 Labelling on Sangaku

In the Greek tradition, points were often labelled using letters. A line could be labelled at its start and end point by characters - say for instance A and B - and line labelled AB. But in the Japanese tradition, figures themselves, whole lines, or segments are labelled. Points to mark the beginning and end of an extension or a position were not indicated. An example of this can be seen in Figure 5.3, which is taken from the *Sanpo Tenzan Shinan*. In the first image, only circles are labelled by placing characters relating to the Chinese calender in their centre. The second diagram has lines labelled in their middle by characters. Also, dashed lines indicate the start and end points of lines. These dashed lines show the points in space where the extension begins and ends, but the whole extension, and not the points, is labelled.

While the Japanese did use an incremental lettering system, when two lines of the same length appear they are often labelled using the same character rather than a new one as a means of indicating they are the same length. For example, if a line AB is one side of an equilateral triangle, we might label the other sides BC and AC in modern times. But all would be labelled AB in the Japanese tradition, with the characters AB appearing in the middle of the line and not at either end.

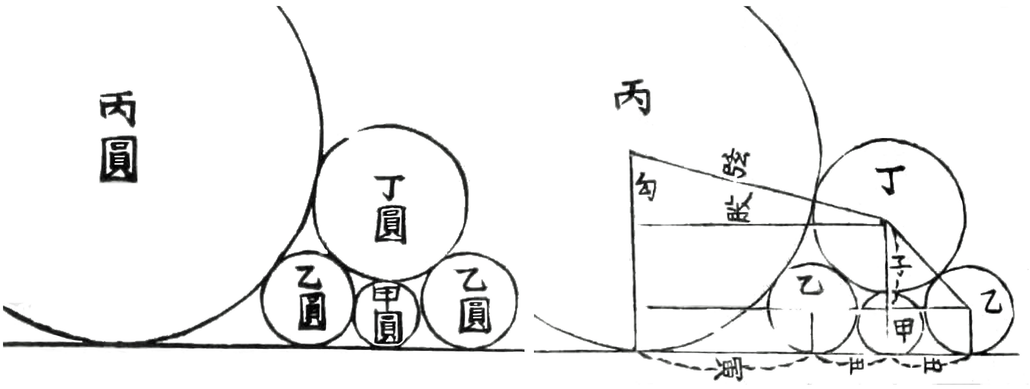


FIGURE 5.3: Left: Problem diagram in *Sanpo Tenzan Shinan*. Right: Diagram with auxillary lines.

5.5.2 Determination of Objects

Concerning this issue of determination, in his investigation of Euclid's *Elements*, Netz contends that most of the time over half the letters on diagrams are under-determined by the text [58, p. 34]. While as mentioned letters to represent points do not feature on Japanese diagrams, a similar question regarding the determination of the labelled figures and lines themselves by the text can still be posed. For instance, if the text says 'there is a line x', is there an actual line labelled x on the diagram?

First to consider with regard to determination are instances on *sangaku* where the text refers to the diameter of a circle without any additional information about where the diameter presents in that circle or how it is labelled. In these instances, Netz argues that including the term diameter "may offer all the determination you need" [58, p. 33] since there is no specific diameter with associated lettering that needs to be determined, and the term diameter can encompass all possible diameters. There are places however where a lack of determination clearly presents on *sangaku*. In fact, most of the *sangaku* problems presented in chapters 2 and 3 are under-determined, with their figures and values not adequately described by the text, meaning they rely on an accompanying diagram to find determination. To illustrate this, consider the following text from a *sangaku* problem not examined in chapters 2 or 3

> There is a circle *zen* 全 containing four circles - large, medium, and small. Say the large circle diameter is known, the medium circle diameter is known, and the small circle diameter is known. Problem - what is the diameter of circle *zen* 全?

In this problem, the text says there are four circles but only gives three labels - large, medium, and small. Since circle *zen* 全 is the circle which contains them all, it cannot itself constitute as this last circle. This means one of these three labels is applied to two circles, but which circle has a double is not stated, and there is no way of knowing from the text alone which it is. Also, while the text gives an indication of their sizing through their labelling (large, medium, small), it does not state the size of each circle in relation to the larger circle *zen*. Do they have a point where they touch the circumference of *zen*? Do they have points where they touch the circumference of the other circles in *zen*? Is the large circle three quarters the size of circle *zen* or half? Is the small circle one third its size or two thirds? Also, what is the arrangement and positioning of the circles? Is the small circle the uppermost, the lowermost, or in the centre? There is no way to determine any of these facts without seeing a diagram.

Another example where a *sangaku* text is under-determined is the Atago tablet (section 2.8.7) where the problem reads (literally)

> As in the diagram, there is an uchiwa fan with a large triangle and small triangle inside. Say the small triangle has a length of 1 *sun*. Problem - what is the big triangle side length?

Here again the text alone is not enough to determine the objects or solve the problem. The author does not state what specific shape the fan is, which is necessary seeing as how uchiwa fans can be perfectly round, slightly oval, or wider at the top than the bottom. Also, where the triangles are placed and their proportion to one another is not known. The only information discernible is that one is larger than the other through the use of the labels 'large triangle' and 'small triangle'. It is not said what kind of triangles they are, or where they are placed. Do they have points where they touch one another and the fan? While the value of a side of the smaller triangles is given, it is not said which side of the triangle this actually is. The language used is also singular, and it is only from viewing the diagram that this text was translated as 'small triangles' instead of 'small triangle' in chapter 2. Without the aid of the diagram, it is unknown whether there are multiple occurrences of each triangle. The text is therefore not only underdetermined, but unable to be properly understood without an accompanying diagram to visually present the configuration of objects intended by the author.

There are also multiple instances on the Suwa tablet where figures are not determined by the text. A particularly telling instance which displays a reliance on diagrams for necessary information can be seen in the following comparison of the third and tenth problems on the Suwa tablet, which read

> Problem 3: As in the diagram, there is an outer circle and three circles *kō* 甲, *otsu* 乙, and *hei* 丙. Say the diameter of *kō* 甲 is 2 *sun*. Problem - what is the diameter of the circle *hei* 丙?
>
> Problem 10: As in the diagram, there is an outer circle containing large, medium and small circles. Say the medium circle diameter is 10 *sun*. Problem - what is the outer circle diameter?

While the two problems seek different values and do not use the same labels, their texts are similar in that they describe an outer circle containing at least three circles. From just the text alone, it is impossible to know the configuration of the circles and their sizes in relation to one another. Are the large circle of the tenth problem and circle *kō* 甲 of the third problem the same size? Is one the length of the radius of the outer circle and the other one quarter of it? Without knowing these facts, exactly which method needs to be applied to find the solution cannot determined.

From the text, it might seem that the two problems are similar and could have the same diagram and solutions. However, as visible in Figure 5.4 where the diagram for the third problem is on the left and the tenth problem on the right, they have very different diagrams and do not use the same geometric principles to find their solutions. In the diagram of the third problem, there is a right angle triangle inscribing the outer circle which was not mentioned in the problem text. This triangle is key to solving the problem, as since the sought after circle is located in one of the arcs created by the inscribed triangle, a calculation to find the sagitta of that particular arc produces the diameter of the circle. While the third problem excludes relevant figures in the text, the text of the tenth problem is also under-determined and fails to provide necessary information about the figure. For instance, there are in fact two large circles, two medium circles, and three small circles. The placement of the circles in the diagram provides information needed for the solution, as the diagram shows a right angle triangle can be formed by connecting the centres of one of each of the large, medium, and small circles.

FIGURE 5.4: Left: Third problem from the Suwa tablet. Right: Tenth problem from the Suwa tablet. (H. Kotera[2]).

The excluding of figures given on the diagram in the text - as seen in the tenth problem - is not isolated to this *sangaku*. Another instance is found on the Yoshifuji Mishima tablet (section 2.8.8). Here two of the sides of a right angle triangle are given, and the concept of the triangle being right angled inferred from the characters used. The text asks the reader to find the side length of an equilateral triangle placed inside the right angle triangle. However, also inscribed inside the right angle triangle in the diagram is a circle labelled *zen* 全, not described in the problem text.

While the feature of under-determination does not occur universally in the tradition, the majority of *sangaku* appear to lack determination; in particular those with abstract geometrical diagrams related to the problem. One explanation for the necessity of diagrams in these cases where abstract figures are dealt with could come from the highly visual nature of the *ōema* tradition *sangaku* appeared from. On *ōema*, a story is told through a painting, meaning it is via visual expression that the intended message of the author is given. In a similar way, the geometrical diagrams on *sangaku* may have served as the main source of information for an audience who were used to communication by visual representation. This could also explain why problems that do not relate to abstract geometry still include paintings and graphical scenes.

[2] See http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html

5.5.3 Interdependency

These examples display how the text of *sangaku* problems were under-determined and crucially relied on the diagram for determination. In fact, in these instances the diagram cannot be envisioned or constructed from the text alone, as there is insufficient information. However, while this investigation has shown that for most *sangaku* the diagram is needed to determine specifics of the problem, the diagram is also meaningless without the text, indicating there is an interdependency between the two. An example which displays this comes from the Tennenji tablet (see Figure 5.5). In this example the author writes "There is a triangle of side length 6 *sun*. Problem - what is the square side length". Again the text is under-determined and there is little hope in recreating the diagram from it, for the author does not say where the square is and if there is more than one. The triangle could be inside a square, or a square inside the triangle. Since the location of the square/s is not known, there is no amount of information that can be gathered from the text to solve the problem. But in this instance, while under-determined the text is also necessary to make sense of the problem, for the diagram alone does not itself provide numerical information and is not drawn correctly. The squares drawn on the diagram are not all of the same size, and the top square has been altered to more of a rectangular shape to make it fit inside the triangle. Without the text stating they are squares, it might be assumed this is a rectangle. Also, from the diagram alone what figure is the focus of the problem cannot be determined. Is it the triangle side length or the square? The text is necessary to establish the problem and clarify what figures the observer is seeing.

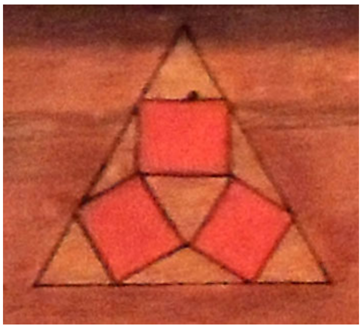


FIGURE 5.5: Problem from the Tennenji tablet. (Image by author).

There are also instances where the diagram itself causes confusion about the problem due to additional elements included which are unnecessary. An example can be seen on the first problem of the Enmanji tablet (section 2.8.6), where a triangle appears with three smaller triangles in its corners and two hexagons drawn in its middle.

The characters for 'problem hexagon' are in the innermost hexagon. However only one hexagon is mentioned in the tablet text, and the observer must assume from the values given in the description which hexagon is the one in question. From the fact that the innermost hexagon is labelled 'problem hexagon', it would seem that this one should be the hexagon used. However the outer hexagon, taking up one third of the triangle side length, seems to fit better given the numerical answer given for the hexagon, which is a third of the triangle side length. Thus the problem is not properly determined with the diagram in this case. This means problems of *sangaku* thus depend on both the text and diagram to function as a problem.

Lastly, on the Suwa tablet, the fifteenth and sixteenth problems are under-determined both by the text and diagram to the extent there is not enough information to solve the problem. The sixteenth problem required the additional assumption that the circles were in an arithmetic progression. There is no way to confirm this is what the author had in mind, for from the text and diagram the individual diameters of the two circles which combined gave 42 cannot be determined. This means in some instances a lack of determination can occur throughout the problem.

FIGURE 5.6: Sangaku with graphical painted scene from Fukui prefecture. (H. Kotera[3]).

5.6 Ornamental Functions and Use of Colour

As shown *sangaku* diagrams can be seen to provide determination and aid in problem solving in certain situations. But as mentioned, some problems do not have geometrical diagrams that accompany them, but graphical scenes and paintings of objects or living creatures. Those which have diagrams also often use beautiful colours to present them. This raises the question of whether diagrams also had an ornamental function to beautify

[3]See http://www.wasan.earth.linkclub.com/fukui/isibe.html

and adorn tablets, as well as an explanatory one. Part of this issue has been discussed in the previous chapter, where it was seen how some *sangaku* contained elaborately painted scenes that make them appear as much as art as mathematics. However, there are geometrical diagrams which also appear to only have an ornamental function. Also, in some cases the choice and use of colour seems to have functioned to highlight figures and their relations to one another, indicating that the use of colour was both ornamental as well as mathematical.

On Kotera's *sangaku* website[4], over 15 percent of tablets contain diagrams which take the form of everyday objects rather than abstract diagrams. The most common object used is a fan, with these accounting for just over 10 percent of the tablets on the website. Honsberger writes that "Figures constructed in a fan (a sector of a circle) became a popular topic of investigation" [30, p. 181] for Edo mathematicians. One example is the Atago tablet (section 2.8.7) where the diagram takes the form of an uchiwa fan. Further examples of fans as *sangaku* diagrams can be seen on the Katayamahiko tablet in Figure 5.9 and the four diagrams in Figure 5.7. There is also a *sangaku* - shown in Figure 5.8 - from the Konno Hachiman shrine in the Shibuya district of Tokyo in which the very wood of the tablet itself is shaped like a folding fan. The problem presented is similar to one seen on the Satimiya tablet (section 3.3.1), but the diagram is presented in a more ornate and aesthetic manner, for it pictures a fine table with spheres placed on its surface.

But what was the importance of fans? It is the case that fans in particular had a highly ornamental value during the Edo and early Meiji periods. The *Japan Encyclopedia* reads

> Their use is widespread in all classes of society...Fans are used by military leaders to direct their troops. Fans are used daily by Shinto priests, Buddhist monks, Noh actors, musicians, sumo referees, and every other social group...Some fans were so beautiful that they were unglued and used to decorate screens. [They] also inspired artists, who made paintings imitating their shape...In 1701, the style of decorated fans had become so elaborate, with some worth a small fortune, that the shogunate promulgated an edict to forbid the manufacture and use of fans that were considered too luxurious [20, p. 186].

[4] www.wasan.jp

[5] Image one: http://www.wasan.earth.linkclub.com/fukusima/fukusimaoji.html. Image two: http://www.wasan.earth.linkclub.com/fukusima/naobi.html. Image three: http://www.wasan.earth.linkclub.com/fukusima/sakai1L2.html. Image four: By author.

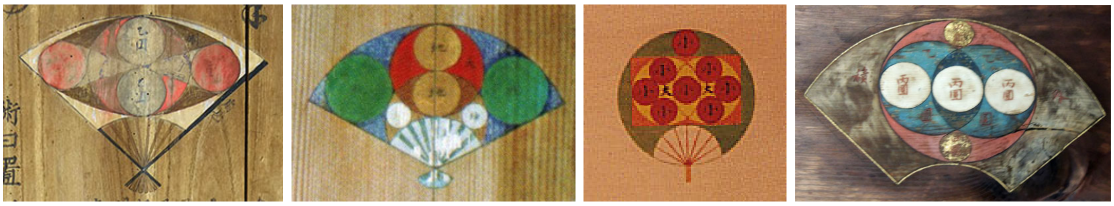


FIGURE 5.7: Diagrams in the form of fans. (H. Kotera[5]).

Displaying problems in the form of a painted fan rather than a circle or sector added a decorative function, for as the *Japan Encyclopedia* indicates fans were connected to art, culture, religion, and status. There are also examples of accompanying images which provide no mathematical information. Two examples include the arithmetic problems discussed in section 5.5.2 which were accompanied by images that were not geometrical in nature. The first of these two problems was accompanied by an image of a snake, and the other by the painted shrine scene in Figure 5.6. These illustrations, while not being diagrams, were purely ornamental and functioned to beautify the tablet rather than provide determination, reasoning, or mathematical information about the problem.

The geometrical diagrams of *sangaku* can also be seen to be ornamental by adorning their problems through the use of colour. In some instances, the colours used are expensive and extravagant. Such paints were mathematically unnecessary, indicating they served a purely aesthetic function. An illustration of this can be seen in the rightmost image in Figure 5.7, where the fan-shaped diagram from an original tablet dedicated in 1844 to Motozenkoji Temple in Iida is elaborately adorned with gold and silver paint. Motozenkoji is a temple of some importance, being the original Zenkoji temple before the temple was moved to Nagano city where it is a main tourist attraction. Through the use of their display as decorative objects related to culture and art in Japanese society, and the expensive, elaborate paints used to present them, the diagrams of *sangaku* in many situations have an ornamental function intended to aesthetically enhance the problem and the tablet in general.

As well as adorning and visually enhancing *sangaku*, colours have an additional function of underscoring mathematical relations and key elements of the diagram. Figures are usually coloured as well as labelled differently when the tablet uses paints. While the use of the colours is completely unnecessary, they still are used for the majority of *sangaku*. For example, Kotera's *sangaku* website also shows over 70 percent of the 450 plus tablets listed contain coloured diagrams. The colours reinforce identifying features such as the labels, and as well as this can indicate equivalences visually and accentuate figures created by the formation of segments or overlapping of figures. This brings attention to aspects of the diagram not necessarily relevant, and may indicate a desire by the creator to make sure all figures - whether used for the problem or not -

are distinguished so the observer can be better aware of all relationships to solve the problem. The author shows their observer everything about the diagram visually - not just what they need to know.

FIGURE 5.8: Sangaku from Konno Hachiman Shrine. (Image by author).

An example of this can be seen in the diagrams relating to fans in Figure 5.7. Here same sized circles are given the same colouring, and different sections of the diagram are painted using different colours again. For example, the ellipse in the leftmost diagram and the rectangle in the diagram second to the far right employ colour as a means to identify the shapes. It can be seen from these examples that colour was used to clearly distinguish between different figures and features of a diagram to determine the relationships between them.

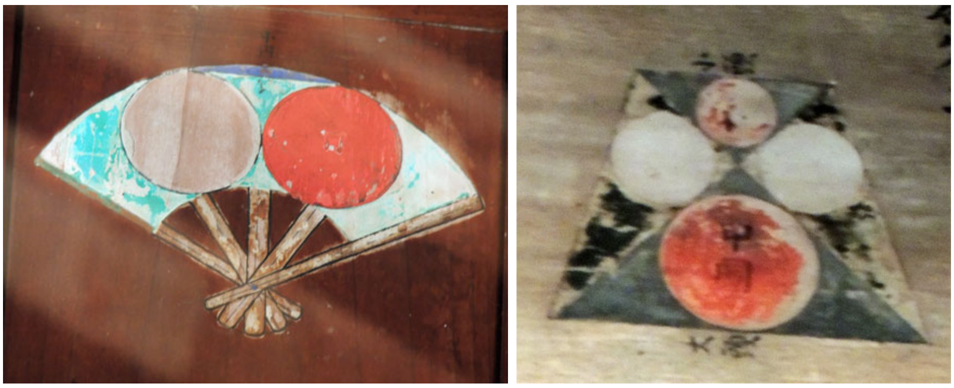


FIGURE 5.9: Left: Fan on Katayamahiko tablet. Right: Sangaku from Nagano prefecture. (Images by author).

However, it is not always the case that similar shapes have similar colours. On the Suwa tablet, there are seven paints and the original wooden background of the tablet used to provide colour inside the diagrams. In the case of the eighth problem (see Figure

5.10), there are three circles and one chord inside a circle. Two of the circles are the same size, but both of these are coloured differently. One of the circles is coloured green, while the other has no colour and displays the brown of the wooden medium of the tablet. The other circle is painted white, as is the chord, while the the circle which contains these figures is painted blue.

Same sized circles being given different colours also occurs in problems 5, 11, 13, 14, 15, 17, 18, and 19 on the Suwa tablet. However, there are still instances where same sized circles are given the same colour. Interestingly, sometimes when there is a set of different sized circles (e.g. two small circles and two medium circles), one set of same sized circles may be coloured differently while the other set is coloured the same. In the case of the fifth problem, one small circle is coloured green, one medium circle coloured red, and then the other small and medium circle are coloured white. Sometimes, similar sized circles are given the same colour, such as in the fourth problem, or all circles are given the same colour regardless of their size, as in the sixth problem.

In problems 4, 11, 13, 15, and 18 the circles whose diameter is sought in the problem have the same colour when there happen to be multiple instances of them in the diagram. One possibility is that this was done to visually differentiate the circles in the problem. But in the eighth problem the two small circles are not the same colour. One possibility as to why this might be is that since only one of the two circles is necessary to refer to in order to solve the problem, and only half the outer circle required, the author is trying to focus on the use of just one of the circles - and not both - to solve the problem. While it is difficult to ascertain the purpose of the author, this distiniction in colours indicates the colouring of different figures and features of diagrams was not arbitrary.

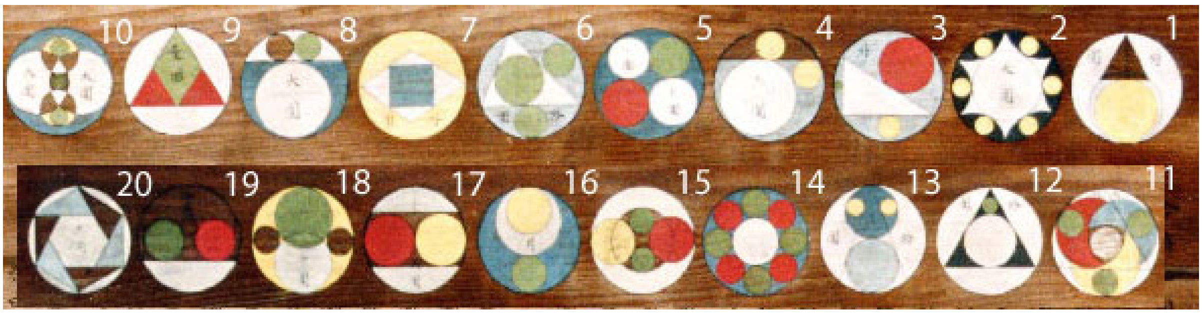


FIGURE 5.10: Diagrams associated with problems 1 to 20 of the Suwa tablet.

Another example of the use of colour to enhance figures on a diagram can be seen in the first image of Figure 5.9. This problem is from the Katayamahiko shrine *sangaku*. It can be observed that all elements of the diagram are coloured, for each circle has its own colour, as well as the fan, its handle, and the arc formed by the chord touching the centres of the top of the two circles. Attention is drawn in particular to the circles and the arc, which use colours that standout against the light blue background. The darker blue of the arc adds emphasis, and helps distinguish it from the rest of the fan as an aspect to be considered in and of itself. Though time has faded the picture, there is also a small circle drawn inside the arc. In the second image on Figure 5.9, a grey colour is used for both isosceles triangles formed by the cutting through of the diagonals in the trapezium, while darker colours are used for the right angle triangles. The two circles in the right angle triangles - which are the same size - are both white, while the two circles in the isosceles triangles are each a different colour. The use of colouring in this diagram shows how similar figures and triangles are distinguished by different colours. Also, the equivalences in the diagram are shown by similar circles having the same colour. As soon as one looks upon this diagram, it is immediately clear what all the elements are through the use of colouring. In this way, while the colours on *sangaku* are technically unnecessary, they are used by the author to make all the parts of the diagram clear and distinct as well as to adorn it.

5.7 Diagrams as Pedagogical, Mnemonic Devices

It has been seen how diagrams had many functions on *sangaku*. They provided determination and were ornamental, but did they also operate as learning mechanisms? When discussing these issues of pedagogy, focus is placed on trying to understand the intentions of the original author and the way they may have facilitated learning in their work [93, p. 11]. Swetz relays that this involves asking if material is sequenced in such a way that simpler problems come before harder problems which build on them, whether there any kind of accompanying instructional dialogue, and if "visual aids; diagrams, illustrations and colours, to assist in the grasping of concepts on the part of the learner" [93, p. 11] appear.

With regard to *sangaku* diagrams, as discussed colours aid in visually distinguishing between different figures and parts of a diagram and often show equivalences and differences. The use of colour could also be considered a pedagogical tool, teaching the observer to recognise differences and equivalences through how they visually are painted. They may also have been used to aid the reader in recognising and remembering which figure was which on the diagram. For, in some instances the diagrams presented have colours but no labels, indicating colours allowed for enough discernment regarding figures that they could be used on their own. However, using "a series of problems. . .arranged in a controlled order to facilitate learning" [93, p. 12] as a pedagogical tool does not seem to be a feature of *sangaku*. In many instances, only one problem is presented on a tablet, and where there are more there does not appear to be any controlled ordering to suggest that each problem builds on from another. Examples of this are the Katayamahiko and Suwa tablets, which have multiple problems looking at different aspects of geometry which do not seem to relate to one another. On the Suwa tablet some problems are similar, but they are not significantly harder or easier in level based on their position.

Setting this aside, the fact that some *sangaku* were created by students may indicate that teachers taught mathematics through diagrams, for this is how the students themselves express the mathematics they have learned. This is further suggested by the use of diagrams in Edo period textbooks dealing with the *tenzan jutsu* technique, where they play a vital role in the solving of the given problems. Figure 5.11 shows diagrams in the *Sanpo Tenzan Shinan* of Ohara connected to the Katayamahiko shrine problem from chapter 3. The original diagram for the problem is on the left, the diagram used to solve the problem in the middle, and an English translation of the middle diagram on the right.

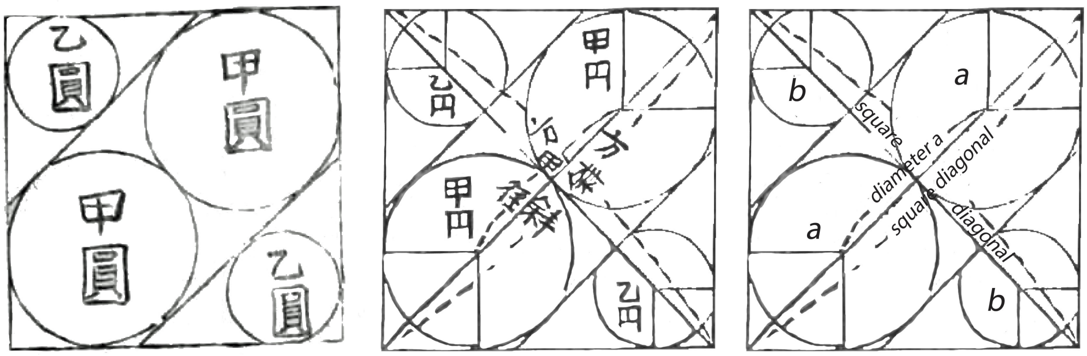


Figure 5.11: Diagrams from *Sanpo Tenzan Shinan*

Observing the diagrams in Figure 5.11, auxiliary lines mark the diagonals of the square. Lines are also drawn which show the radius of each of the circles a and b. From examining the diagram and inserting these lines, it can be seen that the diagonal going from the bottom left-hand corner to the top right-hand corner is made up of the radii of the two circles a, as well as the radii of these circles times $\sqrt{2}$. The square's diagonal is thus $a + \sqrt{2}a$. Looking at the diagonal from the top left-hand corner going down to the bottom right-hand shows the value consists of two of the radii of a, two of the radii of b, and the radii of b times $\sqrt{2}$, giving $a + b + \sqrt{2}b$. In this case from adding the diagonal lines into the diagram and examining how the elements of the diagram relate to them, Ohara shows the lines added can be expressed in different ways which are equivalent. It can be seen that $a + \sqrt{2}a = a + b + \sqrt{2}b$, and from this the algebraic equation $a + b + \sqrt{2}b - a - \sqrt{2}a = 0$ is created which begins the process of solving for the value.

Ohara uses diagrams as tools to teach how a deep examination of the figures and their relationships can be used to derive the answers to conventional geometrical problems. In the cases studies in chapter 3 which used *tenzan jutsu*, it was seen that Ohara constantly referred back to the labelled figures and auxiliary lines he had derived from the diagram. The diagram was therefore a necessary tool for teaching problem solving. Ohara also included accompanying instructional dialogue which had a pedagogic function. It could be argued that the formula section of *sangaku* has a similar function, for it gives instructions for the observer to follow and teaches them the steps necessary to solve the problem. There are also instances in the work of Ohara and also Aida where a sequencing of easier problems to harder does appear, indicating a clear pedagogical imperative in these works.

5.8 Diagrammatic Priority

As shown, in cases where a *sangaku* problem is abstract and geometrical in nature, an accompanying diagram is often necessary to determine the problem text. But was there a priority given to diagrams? When constructing a *sangaku* problem, did the composer specifically begin by drawing a diagram and deriving a problem from it? Or, did they being by writing down a problem and then drawing a diagram relating to it?

There is some evidence to suggest a priority did exist. In many cases on *sangaku*, the author specifically states “As in the diagram” at the beginning of the problem text. This immediately directs the reader towards the diagram and establishes a connection between the two before the problem is described and set. This does seem to point to a priority of diagram over text, as it indicates the text served only as a description of the diagram, with the diagram being the focus and to be considered first. It is also the case that while the Japanese language may not have had adequate terminology to describe all aspects of diagrams, problems could have been described with more detail. For instance, terms indicating location such as left, right, upper, and lower could have been used to help determine placing of circles, and lessen the under-determination. If authors gave priority to the diagram in these instances where we see under-determination, this would explain why the descriptive language for the problems remained so simple, as the text served only to indicate what the problem was, the necessary values, and give the formula used to solve it, with all other information in the diagram. As Larkin and Simon write in *Why a Diagram is (Sometimes) Worth Ten Thousand Words* “If the geometry problem is given verbally, without a diagram, all...elements must be constructed explicitly. . . It is exactly because a diagram "produces" all the elements "for free" that it is so useful" [45, p. 92].

The under-determination of problem texts to the extent that similar text is used to describe vastly different problems suggests also that in many cases the problems were created first visually by means of a diagram and then written secondly to set the problem in simple language. This would indicate that the diagram does not act merely as a heuristic tool or aid for the problem text, but is a key element and source of the problem.

This does not in itself show the diagram necessarily came first in those cases where we have both text and a geometrical diagram concerning it. However, these examples point to a priority of diagram over text. This would mean that since the text is often insufficient to even allow for conceptualisation of the problem, little reasoning can be drawn from it to find the solution. The reasoning for why the problem works, and how it is solved, is drawn from the diagram. For instance, if a circle takes up the full height of an arc, it can be reasoned from this that finding the height of the arc will allow for the diameter of the circle to be found. In the *sangaku* case studies in chapter 3, the *tenzan jutsu* calculations also made explicit references to the diagram. For example, in the first problem of the Mansyouin problem in section 3.3.6, the text 'From picture find all' 依圖求各 appears. This indicates to the observer that certain calculations have been obtained by direct reasoning from the diagram. In this instance, the sides of a right angle triangle are reasoned from the diagram. Examples such as this thus show that diagrammatic reasoning was a crucial part of solving the problem.

5.9 Concluding Remarks

The above analysis of diagrams on *sangaku* illuminates previously unidentified aspects of this tradition. Through examining whether *sangaku* text finds determination through the diagram, an inability to recreate the diagram from the text occurs to such an extreme that the same text can be used to describe different diagrams due to ambiguous wording. This examination has shown that in the Japanese tradition as well as the Greek "part of the content is supplied by the diagram, and not solely by the text" [58, p. 34]. It has also been shown how the diagrams are more likely to have been created prior to the text given what presents at the beginning of each problem, and that there is often a lack of metric precision and issues with proportion. The disconnection metrically and sometimes proportionally of the diagrams provided by the author and that expressed by the text shows the highly abstract form of the diagrams.

Another feature of the tradition highlighted was how colour functions partly as a pedagogic device to illuminate all the aspects of the diagram such that information can be drawn from it easier. These colours also have an aesthetic function, and together with the use of pictures such as fans illustrate an ornamental function of diagrams. Diagrams therefore play a deep, vital role in the *sangaku* tradition, and *sangaku* can be seen as the study of abstract diagrams and the configurations and problems related to them.

Chapter 6

Summary

It has been shown in the previous chapters how *sangaku* should be considered as having communicative, artistic, and religious functions. In chapter 1 I provided a brief overview of the current literature on Japanese History of Mathematics and highlighted the most popular discussions on *sangaku*. Following this, chapter 2 introduced the *sangaku* tradition itself and discussed the history, location, authors, methods, and tools of these tablets. I provided an overview of the language style and numerical units of Edo period mathematics, and also introduced some key terminology. Then a selection of tablets were translated and solved using modern mathematical methods to illustrate what *sangaku* problems looked like and how they are commonly approached in the literature.

Chapter 3 discussed traditional Japanese calculation methods, addressing concerns by Lu that the original form Japanese mathematics took is not shown and discussed in the literature. Chapter 3 also explained the rules of the traditional symbolic manipulation technique *tenzan jutsu* by translating into English parts of the *Sanpo Tenzan Shinan* of Ohara. The traditional solutions given by Ohara for geometrical problems were presented in the form of a transcription of the original calculations, a transliteration, and a modern translation. These problems were discussed in relation to *sangaku*, and used to show how *sangaku* problems could be solved using traditional methods of the Edo period. In chapter 3, exchanges such as that of Fujita Sadasuke and Aida Yasuaki was also used to evidence how *sangaku* functioned to publish and promote mathematical problems.

But while there was a clear purpose to publish and exchange mathematical results, *sangaku* had additional functions as ornamental and religious offerings. Chapter 4 showed how tablets were created using the same materials and artisans as *ōema* offerings, and it was from this very tradition that they emerged. The artisans involved in creating the *sangaku* - once handed over by mathematicians - considered and treated them the same as *ōema*, using the same canvas, tools, and paints. *Sangaku* were also placed in the same areas within shrines and temples as *ōema* - known as *emaden* - which acted as public art galleries during the Edo period. The examples of tablets with non-mathematical paintings, such as the Sozume shrine (Figure 4.6) tablet, and the number of diagrams taking the form of ornamental objects such as fans as seen in chapter 4 also evidence the artistic function of *sangaku*.

Chapter 4 also included examples where the authors of *sangaku* themselves state their intention to dedicate tablets for a religious purpose. One such instance was the student of Takeda, who wrote in the preface to a *sangaku* a wish to ask the Gods to improve their mathematical abilities. There were further religious elements also which presented in the very subject matter of tablets. One example was landscape *sangaku* - such as that from Mizuho shrine (Figure 4.8) - which focused on calculations regarding sacred mountains and religious sites. The very creation of these surveying tablets was due to the religious significance for the author and the wider community of the landmarks examined, for surverying tablets pertaining to non-sacred landmarks do not appear in the tradition. As well as this there was a heavy focus on circular geometry on *sangaku*, which potentially was inspired from circle packing in the lotus from the Buddhist tradition and worship of the sun goddess *Amaterasu* in the Shinto. Therefore, the *sangaku* tradition can be seen to incorporate a rich array of artistic and religious - as well as communicative - functions.

The investigation of diagrams in chapter 5 also showed that when dealing with abstract geometrical problems, the accompanying diagram is necessary to determine the text and the text to determine the diagram. Diagrams however do appear to have prominence and to be the main focus. Under-determination in general is also an issue in the tradition, with some problems - such as the sixteenth problem from the Suwa tablet - requiring additional assumptions to be solved. Some, such as the fifteenth problem of the Suwa tablet, are so under-determined that they are unable to be solved due to lack of information. Diagrams also had ornamental and pedagogic functions on tablets, for sometimes they took the form of non-mathematical objects and used colours to provide information relating to the problem, such as the blue colouring of the small chord in one problem from the Katayamahiko tablet (Figure 5.9) to indicate it was a key element (for its length and sagitta were given and used to obtain the required answer). Colours also aesthetically enhanced tablets, as seen in the use of unnecessarily expensive and ornate paints being used on tablets such as the Mitsu Itsukushima (Figure **??**).

By contrasting the work of Euclid with a *sangaku*, it was also seen how the two styles are very different, and thus terms such as ‘proof’ and ‘theorem’ should be used with caution when describing the Japanese tradition. For *sangaku* authors did not employ diagrammatic or axiomatic proof in the sense of Greek and Western mathematicians.

For as seen, they did not appear to state theorems which were then proved with logical reasoning or argument. On tablets, just a problem, answer, and a formula which provided little information on how it was obtained from the diagram and no reasoning as to why it worked are given. While *sangaku* problems may challenge observers to understand how the they solved the problem, the authors of *sangaku* did not use language indicative of proving or stating theorems.

Thus from this research, it can be seen that the *sangaku* tradition was a branch of traditional Japanese mathematics that functioned to display and transmit mathematical problems in local neighbourhoods and abroad in a way that was sensitive to the artistic and religious environment in which they were placed. They were used by some authors as offerings or prayer requests to the gods, but they were also works of art - though some more deliberately so than others - making them multifaceted, culturally as well as mathematically rich artefacts. They can also be solved using traditional methods of the Edo period, such as *tenzan jutsu*, and constitute a branch or offshoot of the wider *wasan* tradition.

Appendix A

Periods in Japanese History

35,000 – 14,000 BCE	Palaeolithic
14,000 – 300 BCE	Jōmon
300 BCE – 250 CE	Yayoi
250 – 710 CE	Yamato
710 – 794 CE	Nara
794 – 1192 CE	Heian
1192 – 1333 CE	Kamakura
1333 – 1573 CE	Muromachi
1573 – 1603 CE	Momoyama
1603 – 1868 CE	Edo/Tokugawa
1868 – 1912 CE	Meiji
1912 – 1926 CE	Taishō
1926 – 1989 CE	Shōwa
1989 – Present	Heisei

Appendix B

Index Traditional of Books

Jinkoki 塵劫記, Yoshida Mitsuyoshi, 1627.

Sanpo Tenzan Shinan 算法天生法指南, Ohara Toshiaki, 1810.

Sanpo Jikata Taisei 算法地方大成, Hiroshi Hasegawa and Hodo Akita, 1838.

Sanpō Shinshō 算法新書, 1880.

Sanpō Jojutsu 算法助術, Yasunoshin Yamamoto, 1842.

Sanpo Tensei Shinan 算法点竄指南, Aida Yasuaki, 1810.

Appendix C

Tenzan Jutsu Primary Source Material

The case studies in chapter 3 employed instructions using *tenzan jutsu* taken from the *Sanpo Tenzan Shinan*. The original pages which were transcribed and translated in those case studies from this treatise are included in this section.

The original source material used for the Satimiya shrine problems are located in Figures C.1 and C.2, the source material for the Katayamahiko shrine problem in Figure C.3, the source material for the Mansyouin temple problems in Figures C.4 and C.5, and the source material for the Nagano Tenman-gū problems in Figures C.6 and C.7.

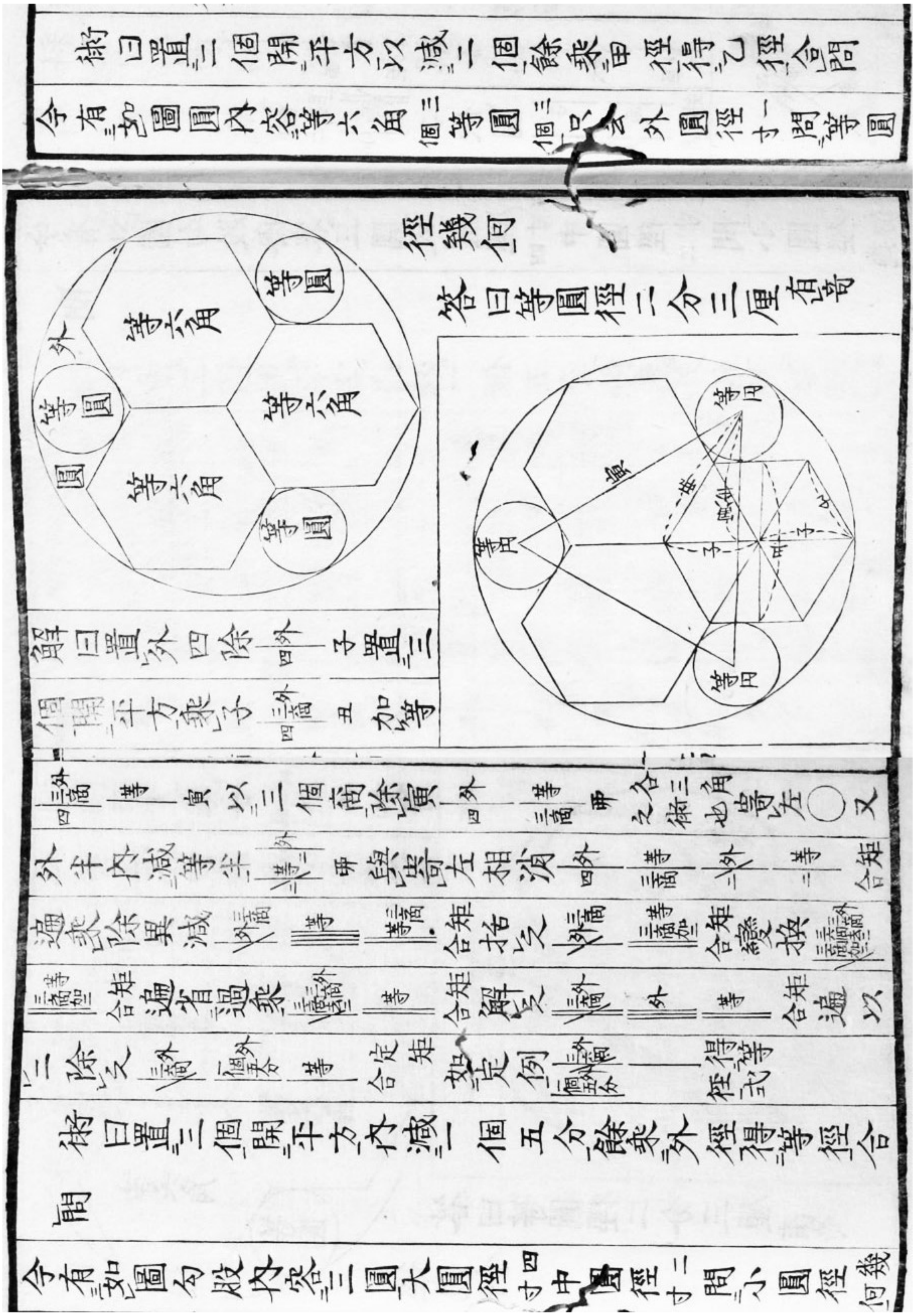


FIGURE C.1: *Sanpo Tenzan Shinan* problem used for the first Satimiya problem.

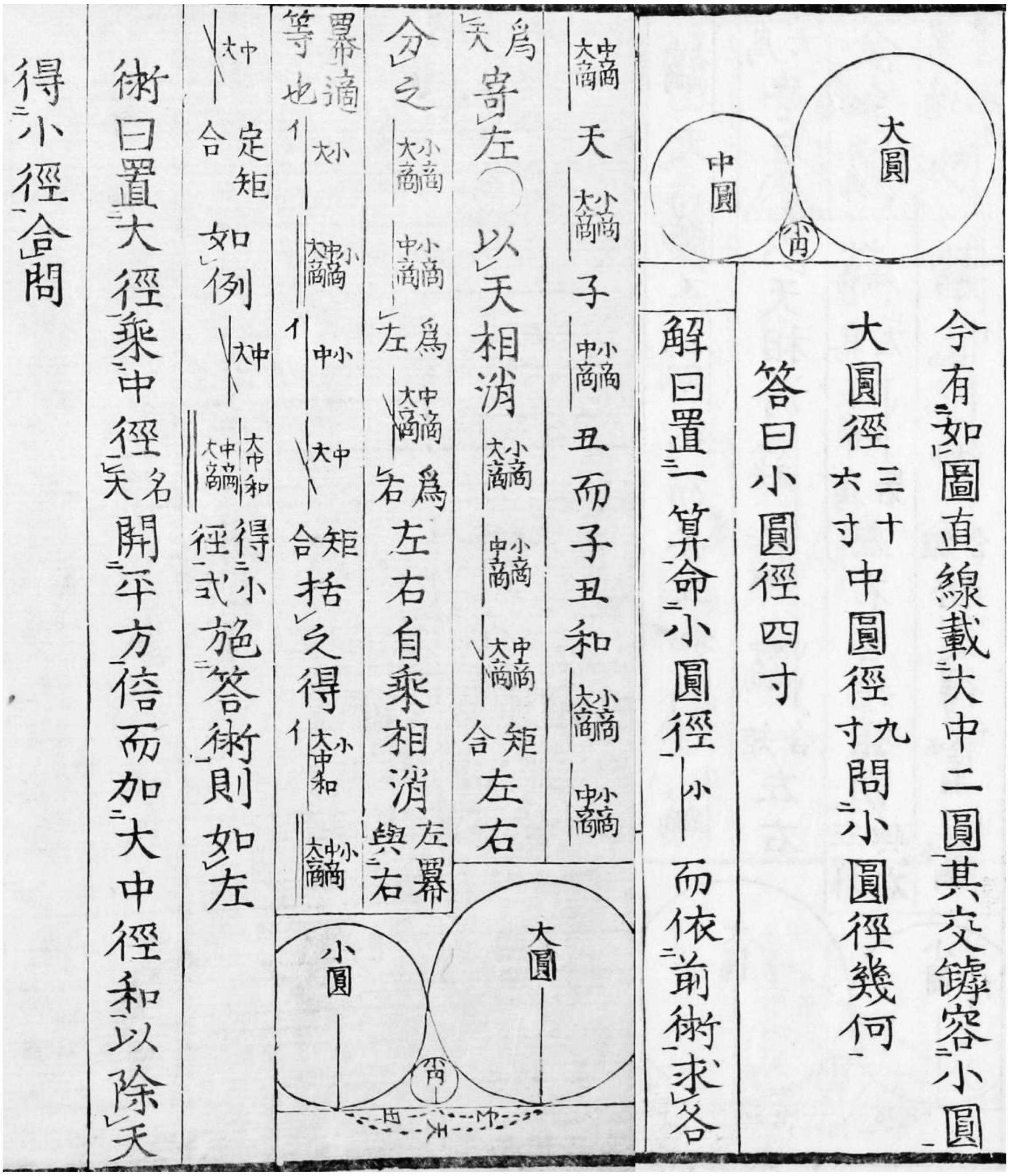


FIGURE C.2: *Sanpo Tenzan Shinan* problem used for the second Satimiya problem.

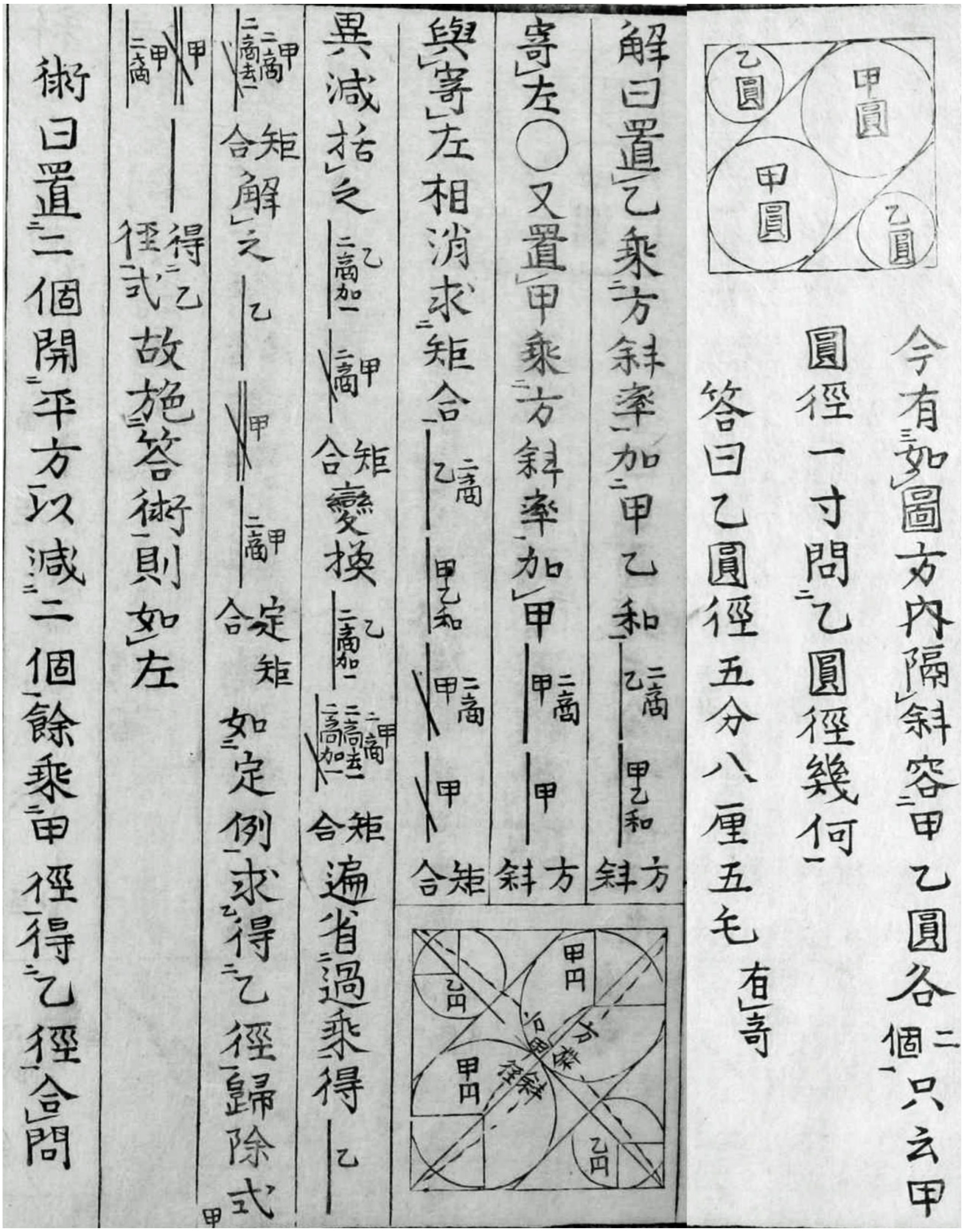

今有如圖方内隔斜容甲乙圓各二個只云甲
圓徑一寸問乙圓徑幾何
答曰乙圓徑五分八厘五毛有奇
解曰置乙乘方斜率加甲乙和
寄左○又置甲乘方斜率加甲
與寄左相消求矩合
異減括之
矩合變換遍省過乘得
矩合解之
定矩合如定例求得乙徑歸除式
得乙徑式故施答術則如左
術曰置二個開平方以減二個餘乘甲徑得乙徑合問

FIGURE C.3: *Sanpo Tenzan Shinan* problem used for the Katayamahiko problem.

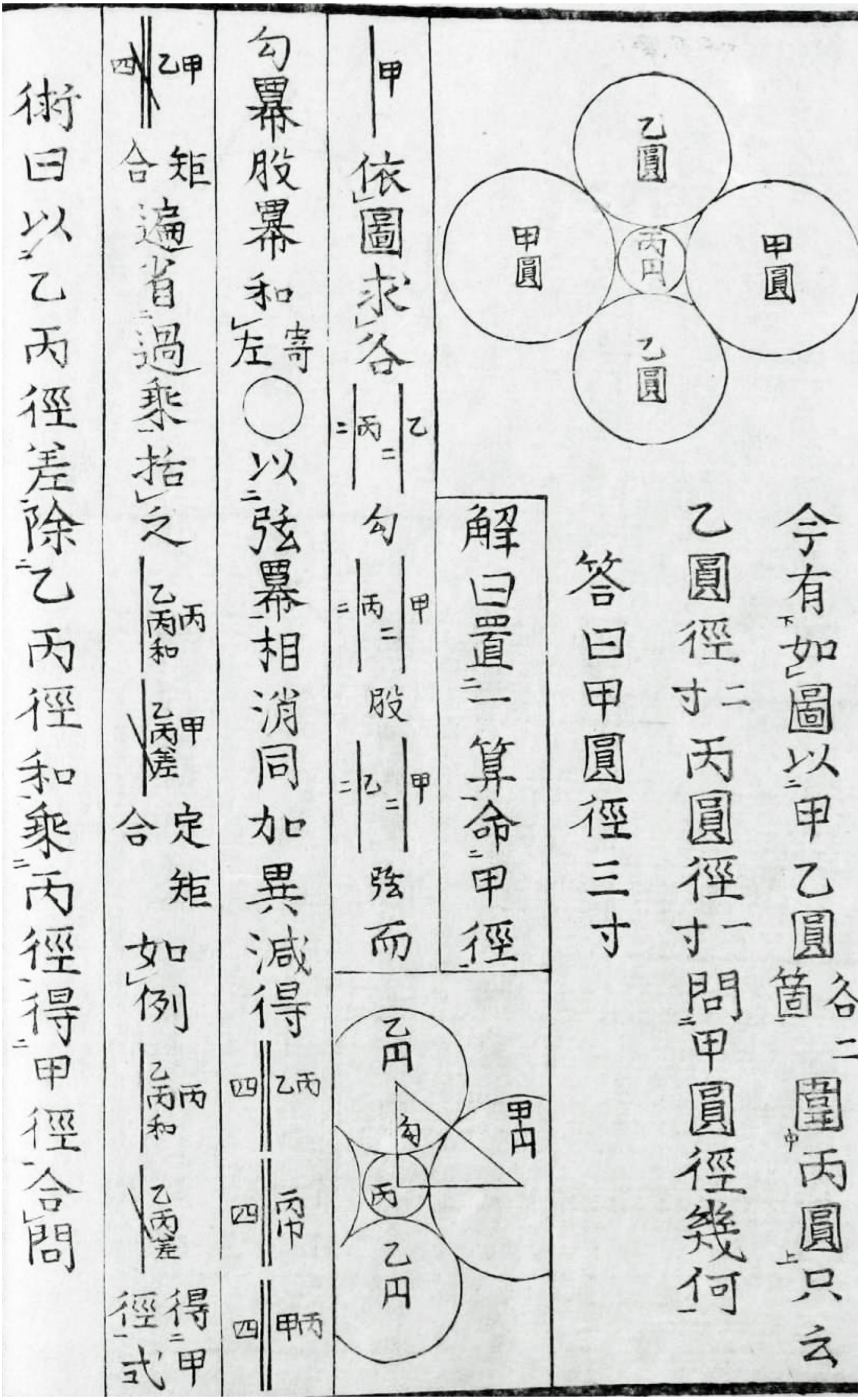

今有如圖以甲乙圓箇各二壹丙圓只云
乙圓徑寸二丙圓徑寸一問甲圓徑幾何
答曰甲圓徑三寸
解曰置一算命甲徑
依圖求各 乙丙 勾 甲丙 股 甲乙 弦而
勾冪股冪和寄左 ○ 以弦冪相消同加異減得 四乙丙 四丙巾 四甲丙
矩合遍省過乘括之 丙乙丙和 甲乙丙差 合定矩如例 丙乙丙和 乙丙差 得甲徑式
術曰以乙丙徑差除乙丙徑和乘丙徑得甲徑合問

FIGURE C.4: *Sanpo Tenzan Shinan* problem used for the first Mansyouin problem.

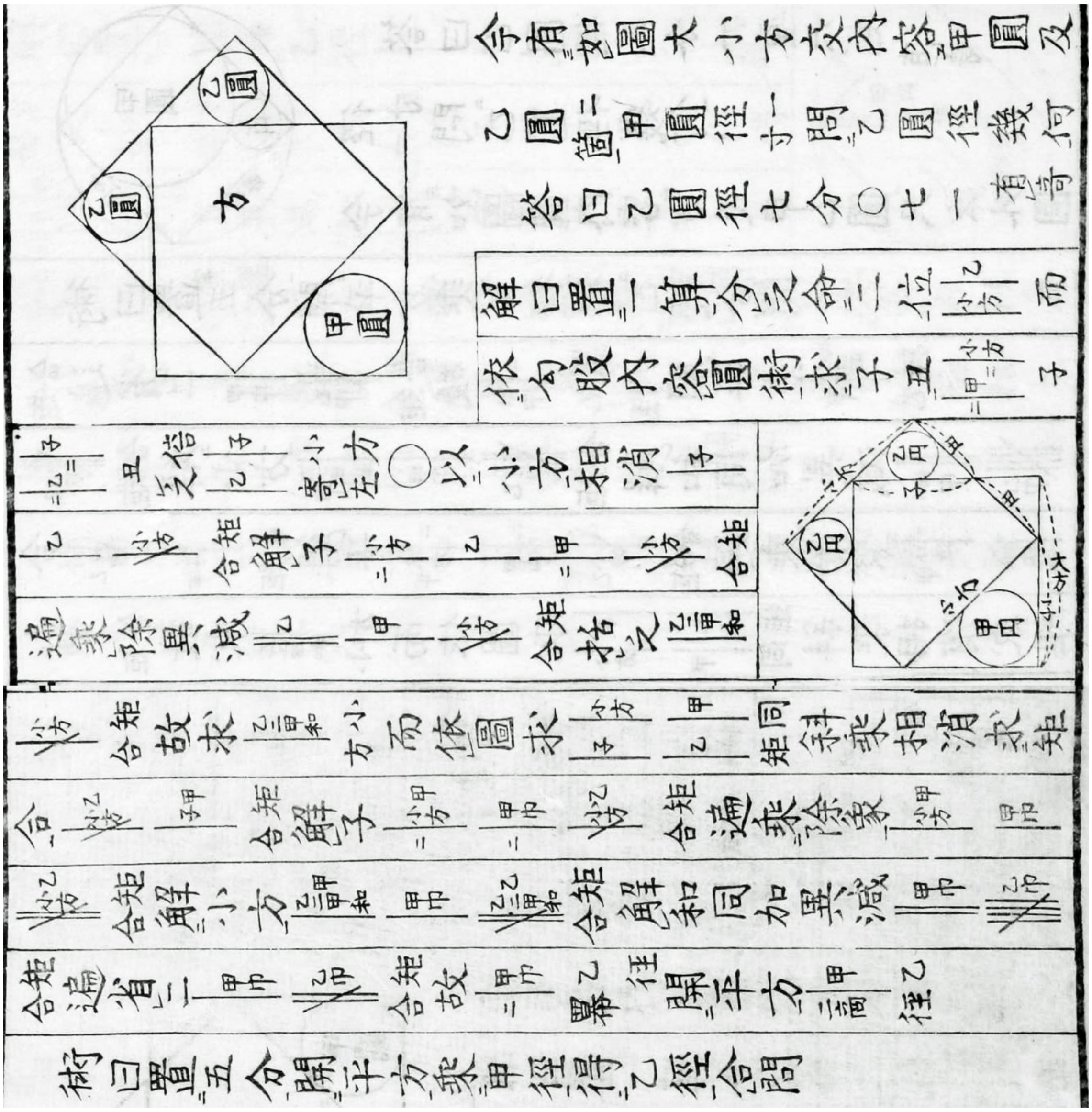


FIGURE C.5: *Sanpo Tenzan Shinan* problem used for the second Mansyouin problem.

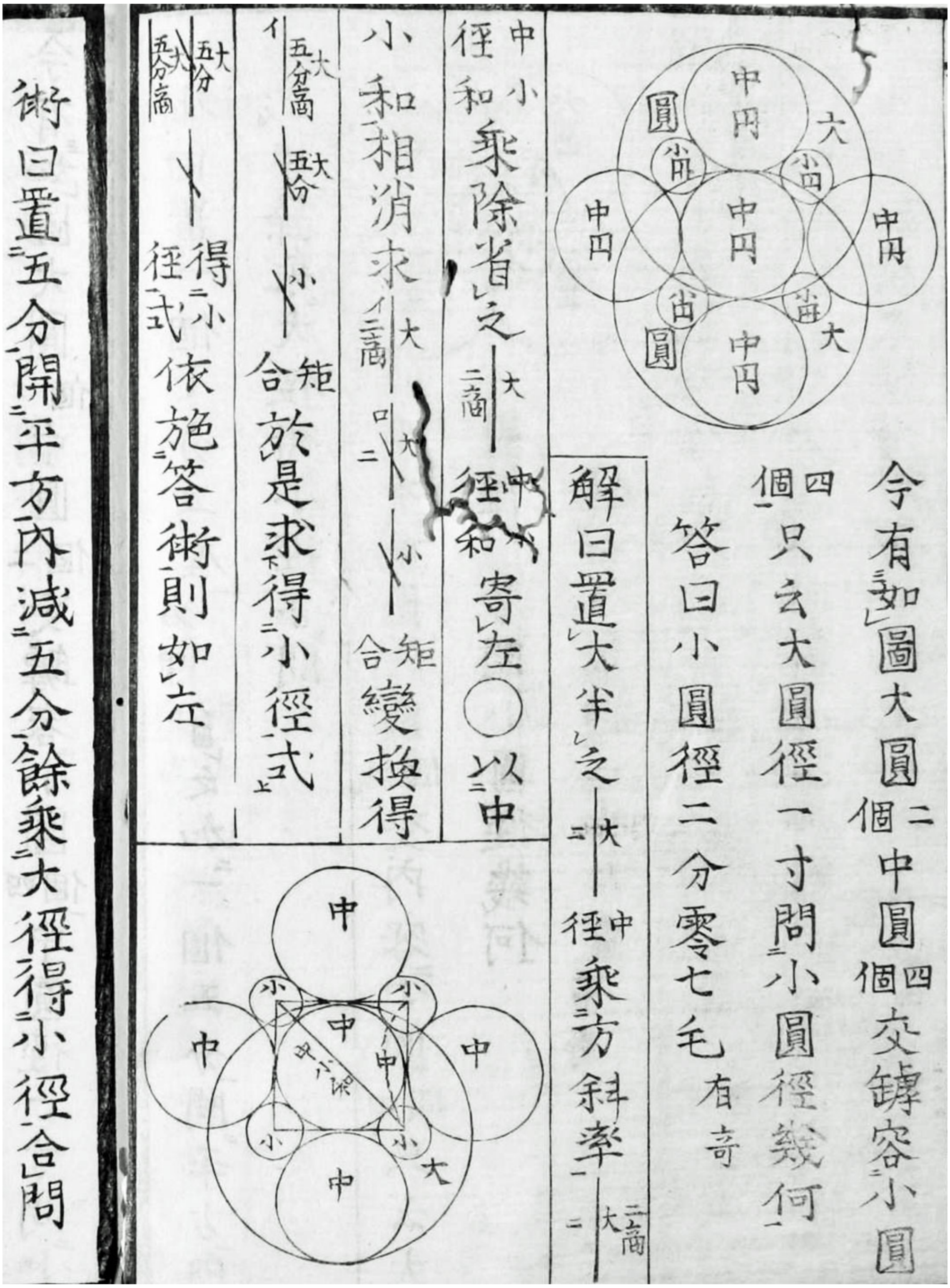

今有如圖大圓二個中圓四個交鑄容小圓
四個只云大圓徑一寸問小圓徑幾何
答曰小圓徑二分零七毛有奇
解曰置大半之 大 中徑乘方斜率 大 二商
中小徑和乘除省之 大 二商 中徑和寄左 以中
小和相消求 大 二商 大 小 矩合變換得
五分大商 五分大 小 矩合於是求得小徑式
五分大 五分大商 得小徑式依施答術則如左
術曰置五分開平方内減五分餘乘大徑得小徑合問

FIGURE C.6: *Sanpo Tenzan Shinan* problem used for the first Nagano Tenman-gū problem.

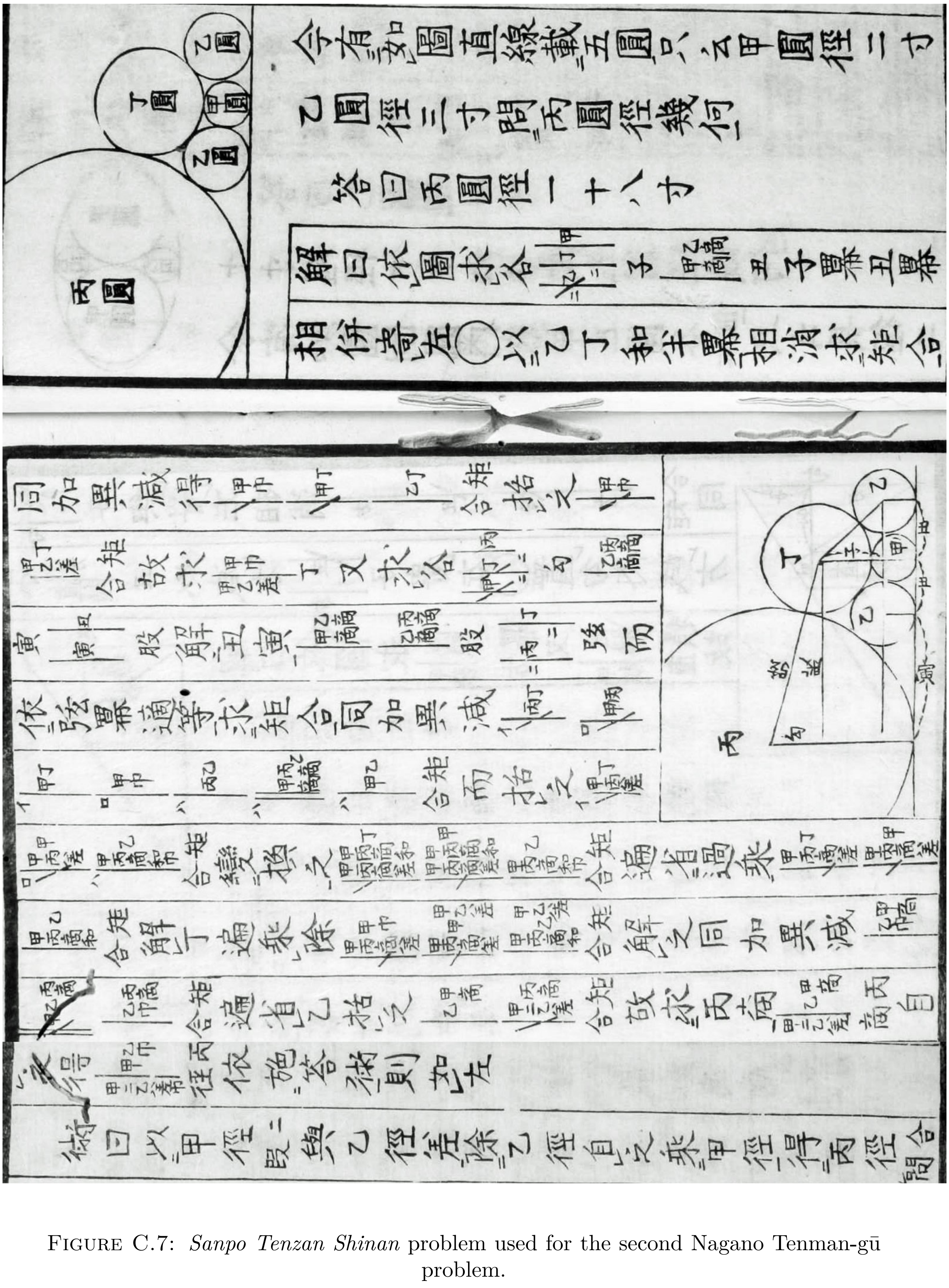


FIGURE C.7: *Sanpo Tenzan Shinan* problem used for the second Nagano Tenman-gū problem.

Appendix D

Sangaku Primary Source Material

A variety of *sangaku* were examined in this thesis. To aid scholars, this section contains larger images of all original *sangaku*. Where the original tablet - or images available of it - are in poor condition, transcriptions which have been used for translations in this section have also been included.

The tablets listed in this section include:

Tablet	Figure
Enman-ji *sangaku*	D.1
Atago *sangaku*	D.2
Yoshifuji Mishima *sangaku*	D.3
Isaniwa *sangaku*	D.4
Isaniwa *sangaku* transcription	D.5
Okiku Inari *sangaku*	D.6
Namigura Inari *sangaku*	D.7
Kumakabuto Arakashihiko *sangaku*	D.8
Kumakabuto Arakashihiko *sangaku* transcription	D.9
Suwa *sangaku*	D.10
Miharu Itsukushima *sangaku*	D.11

TABLE D.1: List of *sangaku* images

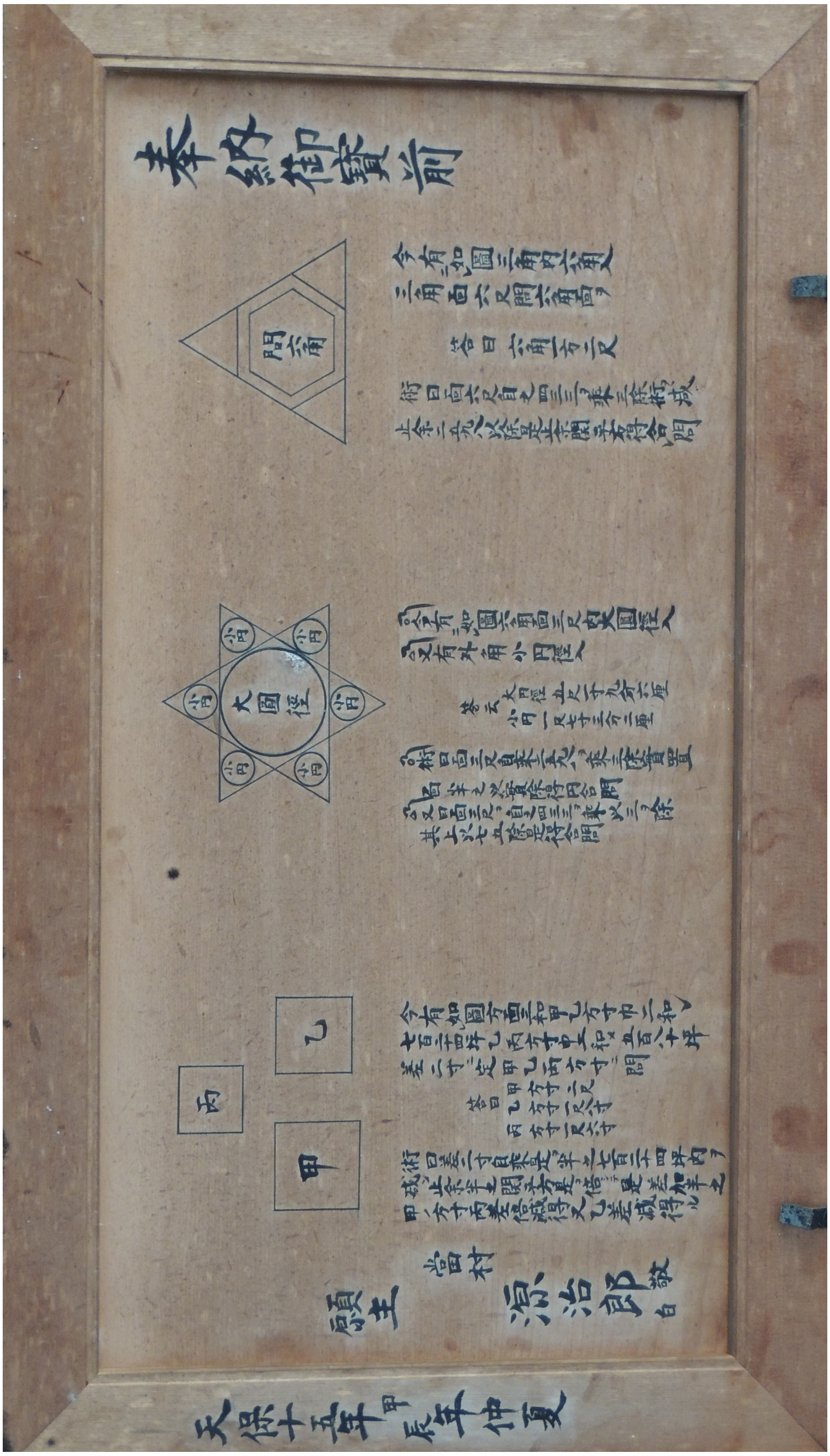


FIGURE D.1: *Enman-ji* tablet. Photograph by author.

Figure D.2: *Atago* tablet. Photograph from H. Kotera (http://www.wasan.earth.linkclub.com/fukusima/atago.html)

FIGURE D.3: *Yoshifuji Mishima* tablet. Photograph by author in 2012.

FIGURE D.4: *Isaniwa* tablet courtesy of the Matsuyama Wasan Society [86, p. 44].

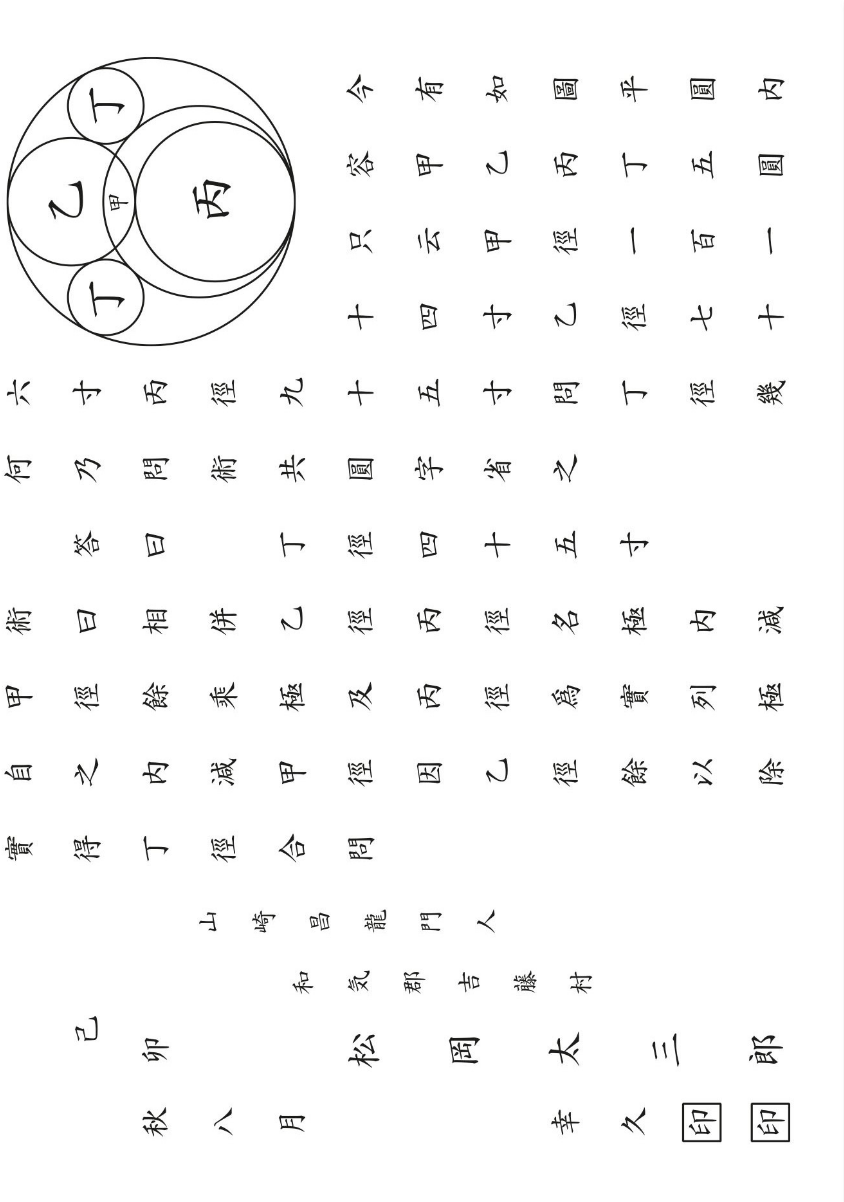


Figure D.5: Transcription of the *Isaniwa* tablet courtesy of the Matsuyama Wasan Society [86, p. 45].

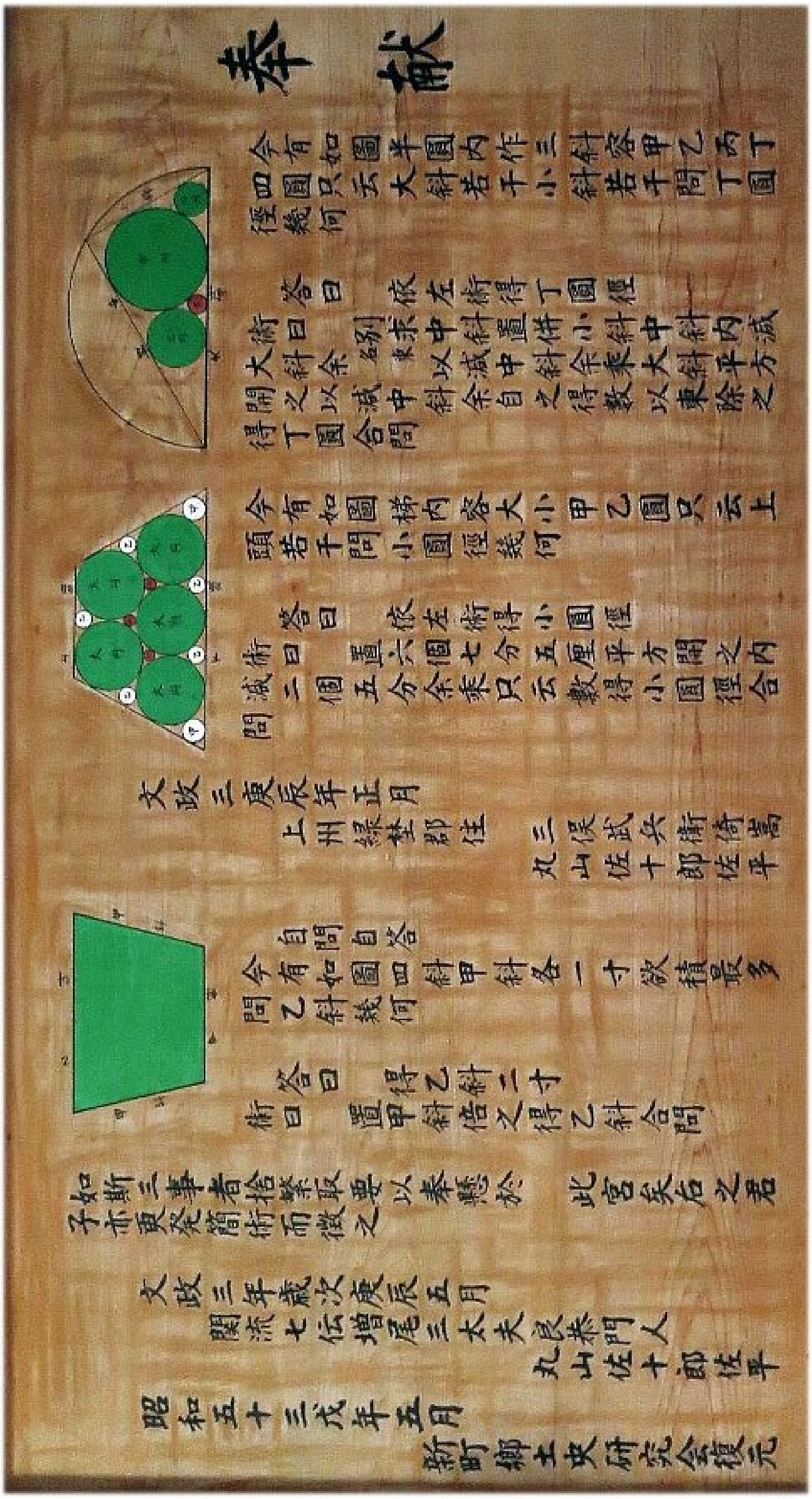


FIGURE D.6: *Okiku Inari* tablet. Photograph from H. Kotera (http://www.wasan.earth.linkclub.com/gunma/okikuinari1.png).

FIGURE D.7: *Namigura Inari* tablet. Photograph by Fukushima University (http://is2.sss.fukushima-u.ac.jp/fks-db/txt/10088.002/image/00019$_0$2.*jpg*).

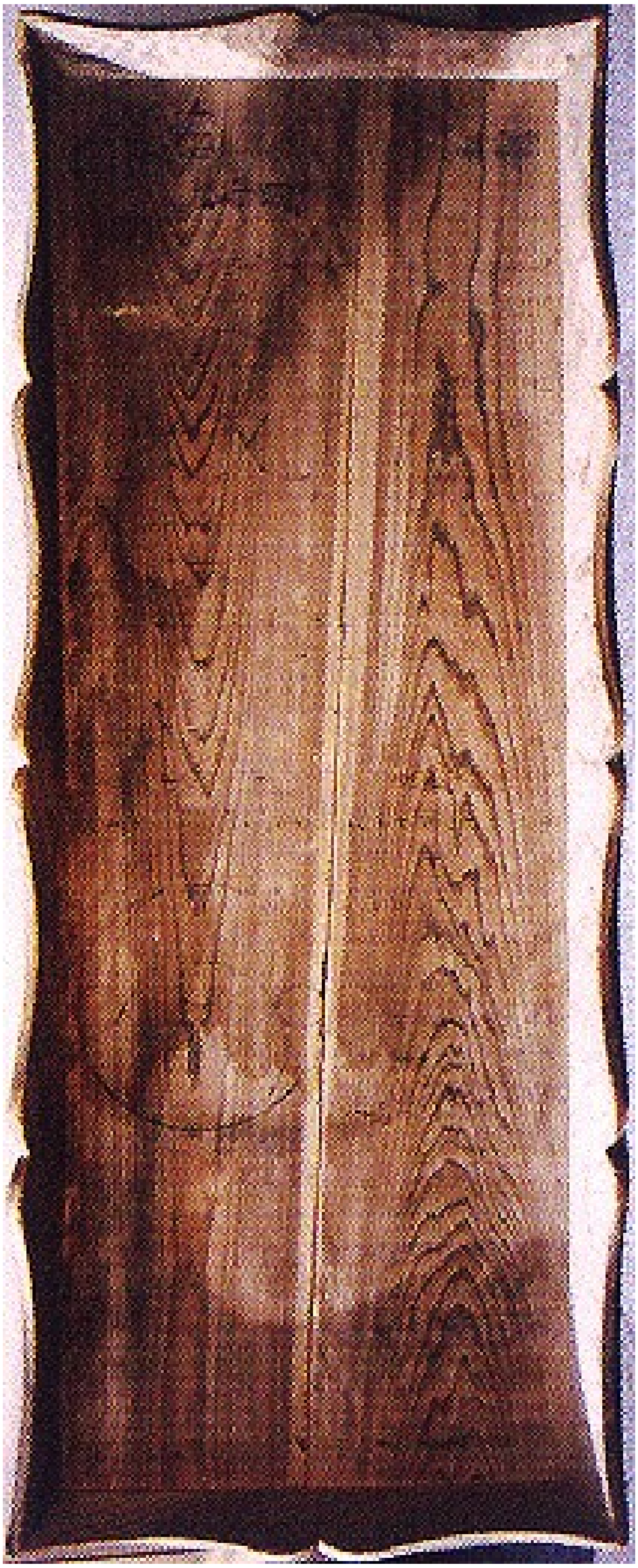


Figure D.8: Original *Kumakabuto Arakashihiko* tablet. Photograph by H. Kotera (http://www.wasan.earth.linkclub.com/isikawa/arakasihiko.html).

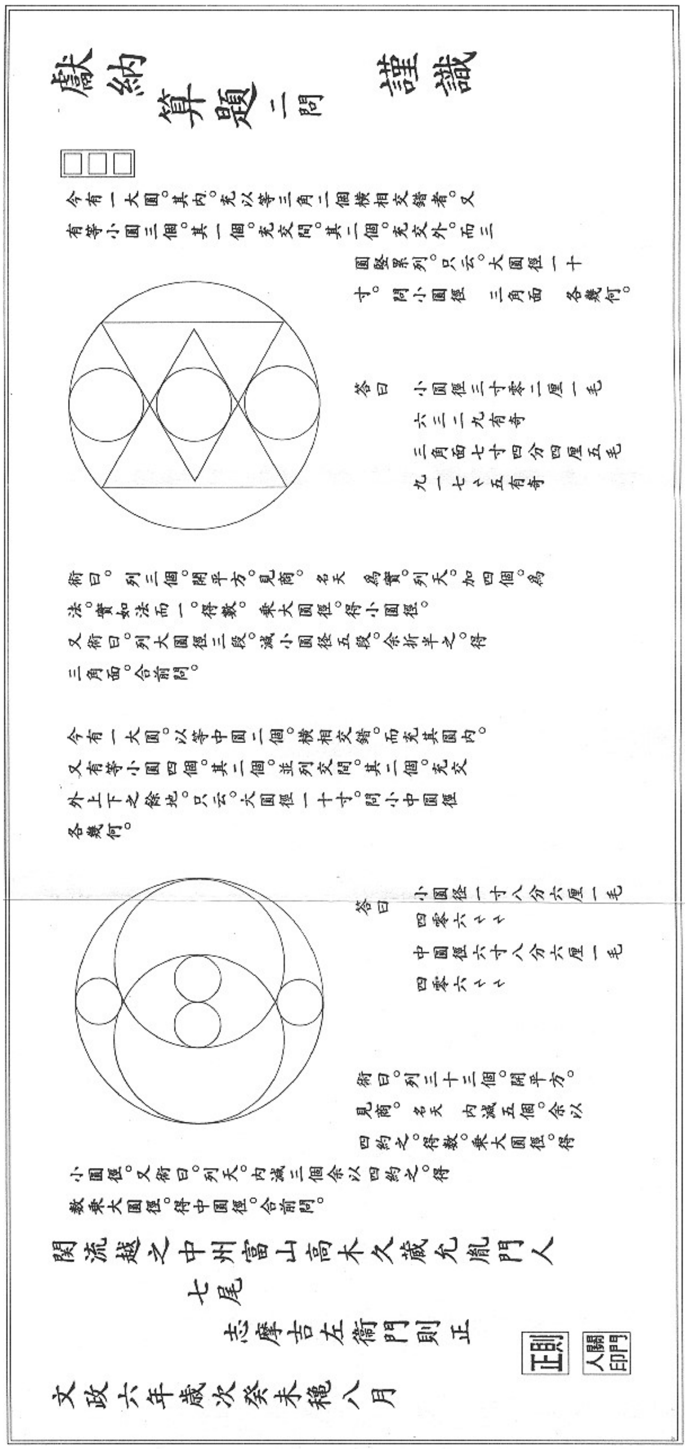


FIGURE D.9: Transcription of *Kumakabuto Arakashihiko* tablet. Photograph by H. Kotera (http://www.wasan.earth.linkclub.com/isikawa/arakasiganbun.html).

FIGURE D.10: *Suwa* tablet. Photograph by H. Kotera (http://www.wasan.earth.linkclub.com/nagasaki/suwa2.html).

FIGURE D.11: *Miharu Itsukushima* tablet. Photograph by H. Kotera (http://www.wasan.earth.linkclub.com/fukusima/miharuitukusima2.html).

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